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Broadway Text-books of Technology . .

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APPLIED MECHANICS

SECOND YEAR

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BY

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APPLIED MECHANICS

CHAPTER I

Acceleration of a Heavy Body.—The experiments described on pp. 99 and 103 of "Applied Mechanics—First Year" reveal the fact that a constant force applied to a motionless object will cause that object to move at a speed increasing uniformly during the time that the force is acting, provided, of course, that the object is not restrained from moving. By interpreting the results of experiments conducted with the apparatus shown in Fig. 1, definite relationships may be established between the quantity of material of which the object consists and the magnitude and time of action of the force. The figure shows a trolly A about 2·5 ft. long, mounted upon light,

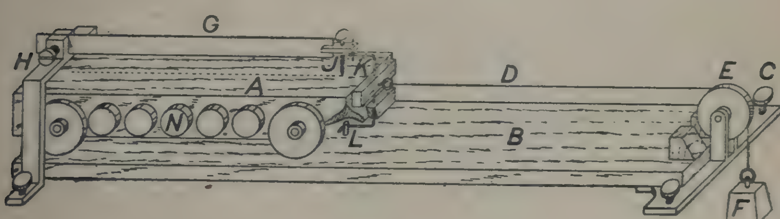

Fig. 1.

freely turning wheels, and supported by a smooth, hard board B rather more than twice as long as the trolly.

A fine silk thread D fastened to the trolly passes around a grooved guide pulley E and supports a small load F. A flat spring, or vibrator, G fixed to a bracket H carries a camel-hair pencil J which, when charged with ink, is capable of tracing lines upon a sheet of paper pinned on the top of the trolly. A thin but fairly stiff piece of steel wire K offers a slight frictional resistance to turning in two small wooden brackets attached to the end of the trolly. The lower part of the wire is bent to form a detent which engages with a small peg L fixed in the board. The upper part of the wire is bent over in such a way that if the spring G is pulled aside it will, when set free, cause another peg at the end of the spring to push the upper end of the wire aside and release the detent. Before release takes place the pull along the cord produced by the load F is balanced by the reaction of the peg L against the pressure of the detent; but when the latter is drawn aside, the pull

along the cord becomes an unbalanced force and causes the trolly to begin moving at the same instant that the pencil J is passing over the centre line of the trolly. Some or all of the cylindrical pieces of metal N are removed when it is desired to reduce the quantity of material in the trolly.

In order that the accelerating effect of the load F shall not be affected by the frictional resistance of the apparatus, the levelling screws—one of which is marked C—are so adjusted that the board has a slight slope toward the pulley E; just sufficient to enable the trolly to roll at a constant speed when the load F is removed and the trolly has been started with a push. The adjustment must be performed with great exactitude to secure a proper effect, and should be tested thus: Obtain a centre line by rolling the trolly along the board away from the pulley whilst the pencil is at rest; set the vibrator in motion and give the trolly a short, sharp push toward the pulley. Ignoring the first part of the curve, traced by

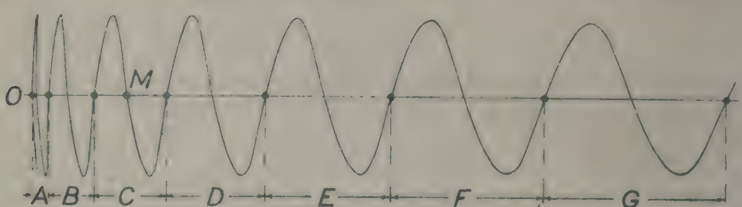

FIG. 2.

the pencil whilst the impelling force was exerted, measure the intercepts of the centre line formed by the rest of the curve. When the adjustment is perfect all these intercepts will be of equal length.

Now attach the load F, obtain a centre line on a fresh slip of paper, and prevent the trolly from moving by means of the detent. When drawn aside and released, the vibrator will push the detent away from the peg and the combined movements of the trolly and the pencil will result in the formation of the curve shown in Fig. 2. The figure is reproduced from a tracing of the line obtained when the weight of the trolly was 7·55 lb. and that of the load F was 0·3 lb. Seven intercepts are shown, but eight were formed. Five complete vibrations per second were made by the spring. The intercept A was formed in the first fifth of a second, the intercept B in the second fifth, and so on. Each intercept was measured and its length expressed in terms of one ft. Each length is equal to the product of the period of time in which the intercept was formed and the mean speed during that interval. Hence, the intercept E being 0·22 ft. long and the time of its formation being 0·2 sec.,

$$\text{mean speed} = \frac{0{\cdot}22}{0{\cdot}2} = 1{\cdot}1 \text{ ft. per sec.}$$

Similarly, the mean speed of formation of each of the other intercepts was found. In this manner the following table was formed:—

Intercept	A	B	C	D	E	F	G	H
Length (ft.)	0·0275	0·078	0·1266	0·1725	0·22	0·2692	0·3192	0·3692
Mean speed of formation (ft. per sec.)	0·137	0·39	0·633	0·862	1·1	1·346	1·596	1·846

The foregoing results are exhibited in the graph B, Fig. 3.

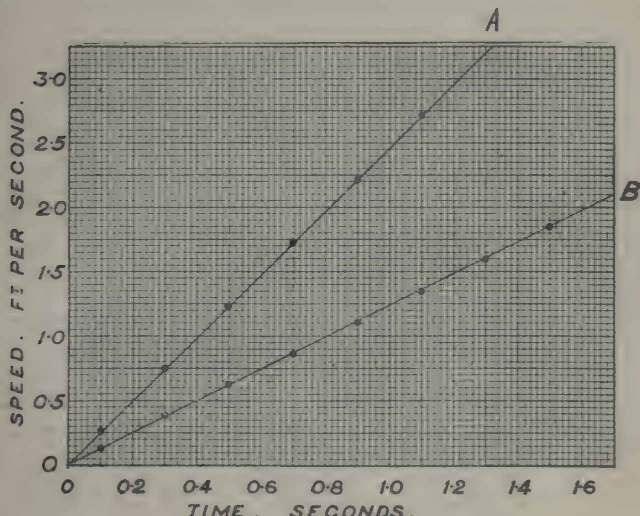

FIG. 3.

Since the speed increases uniformly, each tabulated mean speed coincides with the actual speed of the trolly at the *middle* of the period during which the intercept was formed. In the intercept C, Fig. 2, for example, this instant is indicated by the point M. The pencil on the spring coincided with the point M after the spring had made 2·5 vibrations. Since each vibration occupies 0·2 sec. the mean speed of formation of the intercept C is therefore plotted at the point on the time scale which represents 2·5 × 0·2, or 0·5 sec. after the commencement of the movement. The slope of the graph B, Fig. 3, is 1·234 and represents an acceleration of the speed of the trolly at the rate of that number of ft. per sec. per second.

Load was now removed from the trolly until its weight was reduced to 3·625 lb. Without altering the load F, another curve,

resembling that in Fig. 2, was obtained, and from this the following numbers resulted :—

Intercept	A	B	C	D	E	F
Length (ft.)	0·052	0·15	0·247	0·346	0·4434	0·542
Mean speed of formation (ft. per sec.)	0·26	0·75	1·235	1·73	2·217	2·71

Plotting these numbers gives the graph A in Fig. 3 ; its slope is 2·46. Both graphs in the figure are straight and both pass through the origin ; the latter result being due to the device whereby the commencement of the curve in Fig. 2 is caused to concur with the starting of the trolly.

In both experiments the load F was 0·3 lb. Although this load moved in a vertical direction whilst the trolly moved horizontally, yet the speed and the acceleration of the load along its vertical path were at any moment exactly the same as the speed and the acceleration, respectively, of the trolly along its horizontal path. There is no reason, therefore, why we should not regard all the material composing the trolly and the load F as one piece of substance when considering the acceleration produced by the action of gravity upon the load F. We have accordingly

Graph A :—Material acted upon = 3·625 lb. + 0·3 lb. = 3·925 lb.
Graph B :—Material acted upon = 7·55 lb. + 0·3 lb. = 7·85 lb.

The total quantity of material in the case B is exactly twice that in case A, for in the second experiment the weight of the trolly was carefully adjusted to secure this ratio. Comparing the slopes of the graphs, which are 2·46 for A and 1·234 for B, we find that

$$\frac{\text{acceleration produced in case B}}{\text{acceleration produced in case A}} = \frac{1{\cdot}234}{2{\cdot}46} = 0{\cdot}502 \text{ (nearly)}.$$

Assume that the discrepancy between the last number and the value 0·5 is due to experimental error. Then, since the acceleration of the heavy trolly (B) is half that of the light one (A), and the weight of the material acted upon in the case A is the half of that acted upon in the case B,

$$\frac{\text{acceleration produced (B)}}{\text{acceleration produced (A)}} = \frac{\text{weight of material acted upon (A)}}{\text{weight of material acted upon (B)}}.$$

Mass.—Using trollys of different weight, we have found two different rates of acceleration produced by the same force, viz. : 0·3 lb. Since in each case the weight of the trolly was balanced by the upward reaction of the supporting board it is evident that

weight must be a measure of something more than the pull of gravity.

When a man buys material "by the pound" he is not concerned with the action of gravity; all he thinks of is the quantity of material he requires. Weight, in this instance, is regarded as a measure of quantity of material. Let us accept the same point of view.

Now the weight of the load F has been termed an "unbalanced force." Does this mean that the force in question meets with no resistance? Look at the silk thread whilst the trolly is moving; the absolute straightness of the thread is clear evidence of a state of tension. Tension implies the existence of two equal forces acting in opposite directions. Hence, the moving trolly opposes a resistance equal to the pull caused by the load F.

Resistance to acceleration is due to a property, possessed by all kinds of material, which is termed Inertia or Mass.

This resistance is in no way due to the weight of the piece of substance, for weight is a force which may be neutralised by balancing. In the case of the trolly in Fig. 1, for instance, the weight of the trolly is balanced by the upward reaction of the board upon which it rolls.

From the observations of astronomers it is calculated that the gravitational force of the moon is one-sixth of the gravitational force of the earth. Suppose 1 lb. of substance to be weighed out on the earth, and together with a spring balance—graduated on the earth—to be taken to the moon. Since the extension of a spring is proportional to the force exerted upon it, the substance which weighs 1 lb. on the earth will cause the balance to indicate $\frac{1}{6}$ lb. on the moon. The weight of the substance therefore has diminished to $\frac{1}{6}$ lb., but in spite of this alteration the acceleration produced in the portion of substance is exactly the same on the moon as on the earth when equal forces are applied. This illustrates the fact that the mass of a body, unlike its weight, is independent of position in space and of the influence of other bodies.

The word "mass" is sometimes used to mean the actual substance; sometimes it is defined as "quantity of matter"; but "mass" is to be regarded as a *property* of matter, much as weight, colour, volume, etc., are properties of matter.

Imagine ten similar brass 1-lb. weights standing in a row. Whatever pertains to any one of them pertains to any other. Thus whatever quantity of mass is possessed by one of them is possessed by each of the rest. Hence if all the pieces of metal are melted down into one, the resulting piece of metal will possess ten times the weight and ten times the mass of one of the original pieces. Considering a number of objects at any particular place, either on the surface of the earth or not, their respective weights are proportional to their respective masses. This conclusion is easily arrived at when different quantities of the same material are considered; it is

equally true, however, when the weights and masses of bodies of different materials are compared. The quantity of substance in an object may therefore be expressed in terms of either its mass or its weight.

Unit of Mass.—Mass may be measured and expressed in terms of a unit of mass.

If to a portion of substance a unit of force is applied so that the resulting movement takes place at unit rate of acceleration, then that particular piece of substance possesses unit mass.

Adopting as a unit of force the weight of 1 lb. of matter and as the unit of acceleration 1 ft. per sec. per second, let us determine the quantity of material possessing a unit of mass. Examine the results obtained in the experiments last described. In the case A, a force of 0·3 lb. weight caused an acceleration of 2·46 ft. per sec. per second in a quantity of substance weighing 3·925 lb.

If the force and the quantity of substance are increased in the same proportion, the acceleration will not alter.

Suppose another trolly and hanging weight, each exactly the same as in the case A, say, to be dealt with in the same manner as those actually experimented upon; then whatever happened to one trolly would happen to the other, and they would have a common acceleration. Doubling the quantity of material, therefore, does not affect the acceleration if the force is doubled also. The same is true for any other ratio which applies equally to the mass and the accelerating force. Increasing the force of 0·3 lb. to 1 lb. and the weight of the trolly from 3·925 lb. to Q lb., each in the same proportion, we have

$$0{\cdot}3 : 1 :: 3{\cdot}925 : Q$$

$$\therefore \quad Q = \frac{3{\cdot}925}{0{\cdot}3} = 13{\cdot}083 \text{ lb.}$$

A force equal to the weight of 1 lb. will therefore produce an acceleration of 2·46 ft. per sec. per second in 13·083 lb. of matter. In what quantity of matter will a force equal to the weight of 1 lb. produce an acceleration of 1 ft. per sec. per second?

The last equation on p. 4 derived from the experimental results shows that if the quantity of matter is halved, whilst the accelerating force remains unchanged, the acceleration produced will be doubled. The general relationship illustrated by the equation is this:

The acceleration produced by a constant force is inversely as the quantity of matter acted upon.

Let m denote the number of lbs. of matter which, when acted upon by a force equal to the weight of 1 lb., acquires an acceleration of 1 ft. per sec. per second. Since the same amount of force, the

weight of 1 lb., produces an acceleration of 2·46 ft. per sec. per second in 13·083 lb. of substance,

$$\frac{\text{acceleration produced in } m \text{ lb.}}{\text{acceleration produced in } 13{\cdot}083 \text{ lb.}} = \frac{1 \text{ ft. per sec. per second}}{2{\cdot}46 \text{ ft. per sec. per second}}.$$

$$= \frac{13{\cdot}083 \text{ lb.}}{m \text{ lb.}}$$

$$\therefore \quad m = 13{\cdot}083 \times 2{\cdot}46 = 32{\cdot}18 \text{ lb.}$$

Hence,

A piece of matter which weighs 32·18 lb. will receive an acceleration of 1 ft. per sec. per second when acted upon by an unbalanced force equal to the weight of 1 lb.

In accordance with the definition of mass on p. 6 this quantity of matter will possess a unit of mass.

Since British engineers are accustomed to use the weight of 1 lb. as a force unit and to express acceleration in terms of feet and seconds, this unit is often termed the "engineers' unit of mass." Even the most refined experiments yield a result which differs very slightly from the one we have obtained, and it is usual to take 32·2 lb. as a sufficiently close approximation.

The quantity of matter which weighs 1 lb. is sometimes regarded as possessing a unit of mass. To correspond with this, a force unit is then chosen which will cause acceleration at the rate of 1 ft. per sec. per second in 1 lb. of matter. The magnitude of the required force unit is thus determined: A force equal to the weight of 1 lb. produces unit acceleration in a quantity of matter which weighs 32·2 lb. If the force and the amount of matter are increased or decreased in the same proportion the acceleration does not alter. Hence if F lb. be the required force,

$$\frac{\text{matter weighing } 32{\cdot}2 \text{ lb.}}{\text{matter weighing } 1 \text{ lb.}} = \frac{\text{force of } 1 \text{ lb. weight}}{\text{force of F lb. weight}},$$

$$\therefore \quad \mathrm{F} = \frac{1}{32{\cdot}2} \text{ lb.}$$

This unit of force, the weight of F lb., that is, is called a *poundal*, and does not differ materially from the weight of $\frac{1}{2}$ oz.

In the metric system a cubic centimetre of water is regarded as possessing unit mass, and unit acceleration is expressed as 1 cm. per sec. per second. The force which produces unit acceleration when acting upon the quantity of matter possessing unit mass is called a *dyne*, and is adopted as a force unit.

A force unit which results in the production of unit acceleration in a body possessing unit mass is termed an absolute unit of force.

The Acceleration of Gravity.—Instead of maintaining a constant accelerating force and varying the mass of the material upon which it acts, the mass may be kept constant and the magnitude of the force varied. In carrying out an experiment in accordance with these conditions, a number of small pieces of metal may be fixed —by drawing-pins, say—to the trolly (Fig. 1), and a very small load used at first to produce acceleration. After obtaining a curve similar to that in Fig. 2, one of the small pieces of metal is removed from the trolly and added to the falling load. A second curve is then obtained.

Proceeding in this way until all the small pieces have been removed and added to the load which causes acceleration, a series of curves results, each yielding numbers which are plotted as

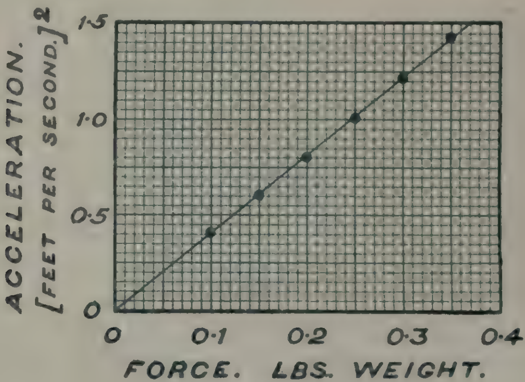

FIG. 4.

described on p. 3. The slope of each graph represents an acceleration. An experiment thus performed afforded the following results; the weight of the trolly, including that of the pieces of metal attached and the falling load, was 7·9 lb.

Falling load (lb.) . . .	0·1	0·15	0·2	0·25	0·3	0·35
Acceleration produced (ft. per sec. per second) . . .	0·402	0·604	0·802	1·02	1·22	1·43

The graph of these numbers is shown in Fig. 4. Along the vertical scale of the graph, instead of "ft. per sec. per second," we find [ft. per second]², which is an alternative method of expression. Note that the *word* "acceleration" is not included in the brackets.

From the graph it is obvious that: *The acceleration produced in a constant mass is proportional to the force acting upon the mass.*

The slope of the graph is 4·07, and expresses the acceleration

produced in a body weighing 7·9 lb. by a force equal to the weight of 1 lb. Suppose the accelerating force to be augmented until it is equal to the weight of the body acted upon; then in accordance with the statement just made the acceleration will increase proportionately. Hence the acceleration produced in a body weighing 7·9 lb. by a force equal to its own weight is $7·9 \times 4·07$ ft. per sec. per second. The product of these numbers is approximately 32·15. If an object weighing 7·9 lb. were allowed to fall freely, the accelerating force acting upon it would actually be its own weight, so that the acceleration produced by the action of gravity would be 32·15 ft. per sec. per second.

The weight of any other body may be denoted as 7·9 K lb. Such a body would possess K times the mass of the one weighing 7·9 lb. The acceleration will remain unaltered because the accelerating force of gravity and the mass acted upon have both been altered in the same ratio. Consequently all falling bodies, of whatever weight or mass, receive an acceleration of 32·15 ft. per sec. per second if the resistance of the atmosphere is negligible.

By refined methods of experiment a slightly higher value than 32·15 is obtained for the acceleration of gravity. Just as "π" is adopted to represent the ratio of the length of the circumference of a circle to that of its diameter, so the letter "g" is used to denote the acceleration due to gravity. Just as the number 3·1416 is taken as a working approximation for π, so the number 32·2 is used in arithmetical calculations where g is involved. If instead of lb. and ft. units of some other kind are employed, the number representing g will not remain the same. In the metric system the centimetre is the unit of length, and 2·54 cm. are equivalent to 1 in. Working with metric units, therefore,

$$g = 32·2 \times 12 \times 2·54 = 981 \text{ cm. per sec. per second.}$$

A block of iron is raised to a height of 40 ft. by a machine for driving piles. What speed will the block attain if it is allowed to fall back to the position from which it was lifted?

As the block falls it is subjected to an acceleration of g ft. per sec. per second. When an object subjected to an acceleration of a ft. per sec. per second moves through a space of S ft. and attains in consequence a speed of V ft. per sec.,

$$V^2 = 2aS.$$

Substituting the value of g for a and 40 ft. for S,

$$V^2 = 2 \times 32·2 \times 40 = 2576.$$

$$V = \sqrt{2576} \text{ (ft. per sec.)} = 50·7 \text{ ft. per sec.}$$

The reader will scarcely fail to notice how closely the number obtained for g approximates to that which was found to represent

the weight in lbs. of the quantity of material possessing an engineers' unit of mass. That they are in truth identical is argued thus: On p. 7 it was found that a body weighing 32·18 lb. possesses a unit of mass; that is to say, a unit of force equal to the weight of 1 lb. will impart unit acceleration to a quantity of substance weighing 32·18 lb. The graph (Fig. 4) shows that an accelerating force and the acceleration produced by its action upon a constant quantity of mass increase in the same ratio. For a body weighing 32·18 lb. (and possessing unit mass), the 1 lb. force which produces unit acceleration must therefore be increased to a force of g lb. weight in order to impart g units of acceleration.

If the body weighing 32·18 lb. is allowed to fall freely, g units of acceleration will result, and the accelerating force of g lb. is its own weight. Consequently g is identical with the number of lbs. weight of an engineers' mass unit. It is usual, therefore, to express the number of engineers' mass units possessed by a body weighing W lb. as $\frac{W}{g}$. It is true that the weight of the body possessing unit mass, viz. 32·18 lb., does not agree precisely with the number obtained for g, viz. 32·15, but the slight difference is accounted for by experimental error.

Force and Acceleration.—It has now been established that if an object weighs W lb., $\frac{W}{g}$ is the number of mass units it possesses, and that a force of 1 lb. weight is required to produce unit acceleration in each mass unit. $\frac{W}{g}$ is therefore the number of lbs. weight required to produce unit acceleration in a body weighing W lb. Since the accelerating force acting on the object and the acceleration thereby produced increase in the same ratio (Fig. 4), then in order to produce a units of acceleration, the accelerating force of 1 lb. weight acting upon each mass unit must be increased a times. That is, to produce a units of acceleration in a body weighing W lb. the accelerating force must be equal to $\frac{W}{g} \times a$ lb. weight. Denoting the force as F lb., then

$$\text{F lb.} = \frac{W}{g}a.$$

This equation expresses a fundamental relationship which must on no account be lost sight of, viz.—

The product of the number of mass units and the number of units of acceleration equals the number of force units whereby the acceleration is produced.

A condensed version of the preceding statement, convenient for memorising, is

$$\text{force} = \text{mass} \times \text{acceleration}.$$

During the first part of its lift, a hoist is accelerated at the rate of 8 ft. per sec. per second. The hoist, together with its load, weighs 6,000 lb., and is supported by a steel rope. What is the tension in the rope during the acceleration of the hoist?

$$\text{Force to produce acceleration} = \frac{W}{g}a = \frac{6{,}000 \times 8}{32{\cdot}2} = 1{,}490 \text{ lb.}$$

$$\begin{aligned}\text{Tension in the rope} &= \frac{\text{force to produce}}{\text{acceleration}} + \text{weight of load}\\ &= 1{,}490 \text{ lb.} + 6{,}000 \text{ lb.}\\ &= 7{,}490 \text{ lb.}\end{aligned}$$

When the metric system of units is employed, the equation expressing the relationship between force mass and acceleration assumes the form

$$\text{dynes} = \text{grams} \times \text{centimetres per sec. per second.}$$

If a body weighing 1 lb. is regarded as possessing a unit of mass, the corresponding force unit is a poundal, and then

$$F \text{ poundals} = W \text{ lb.} \times a \text{ ft. per sec. per second.}$$

Momentum.—Let us now seek for a relationship between the mass of a body, a force acting on the body, and the speed which the body will consequently attain in a given time. On p. 4 is a table showing the speed attained after stated intervals of time by a trolly weighing 3·625 lb. (case A) when subjected to the action of an accelerating force of 0·3 lb. On p. 3 is another table giving similar results when the same force is caused to accelerate a trolly weighing 7·55 lb. (case B). The total mass, $\frac{W}{g}$ lb., acted upon includes that of the load F (Fig. 1).

For case A it is $\frac{3{\cdot}925}{32{\cdot}2}$, or 0·122 mass units.

For case B it is $\frac{7{\cdot}85}{32{\cdot}2}$, or 0·244 mass units.

Let the speed of either trolly at any instant be denoted V ft. per sec. Taking values of the speed from the tables referred to, let

each be multiplied by the corresponding number of mass units. Together with the interval of time required to attain each speed the products in question are tabulated as follows—

Time (secs.)	0·1	0·3	0·5	0·7	0·9	1·1	1·3	1·5
$\frac{W}{g}$ V (case A)	0·0317	0·0915	0·151	0·211	0·27	0·33	—	—
$\frac{W}{g}$ V (case B)	0·0334	0·0952	0·154	0·21	0·268	0·328	0·39	0·45

In Fig. 5 both sets of products are plotted, the points obtained in case A being represented by crosses, those in case B by circles filled in with ink. It is apparent that the plotting of the two sets

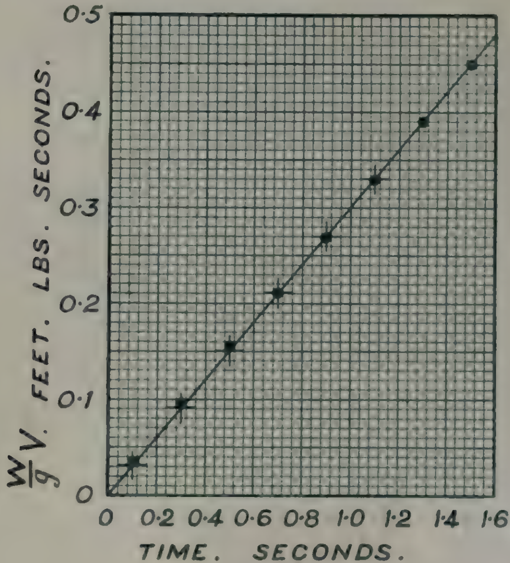

FIG. 5.

of numbers results in the formation of a single straight-line graph. The slope of the graph, 0·3, represents the increase per sec. of the product $\frac{W}{g}$V. The accelerating force in each instance was 0·3 lb. and as this number equals the slope of the graph, it points to the conclusion that the two numbers are necessarily equal.

Let T represent the number of seconds in which the trolly attains a speed V. The slope of the graph (Fig. 5) may then be expressed as

$$\frac{W}{g} V \div T, \text{ or } \frac{W}{g} \times \frac{V}{T}.$$

If a is the acceleration, the speed is increased at the rate of a units of speed per sec.

Starting from a state of rest, in T seconds the trolly attains a speed V which is equal to $a \times T$. Hence,

$$a = \frac{V}{T}.$$

On p. 10 will be found the equation

$$F = \frac{W}{g} a,$$

F representing accelerating force in lb. weight, W the weight in lb. of the body acted upon, and a the acceleration produced. Substituting the value obtained for a,

$$F = \frac{W}{g} \times \frac{V}{T}.$$

As the expression on the right-hand side of the last equation has been found to equal the slope of the graph (Fig. 5), then F, the number of lb. weight of force acting on the mass $\frac{W}{g}$, also expresses that slope. The equation shows why the two sets of products gave only one graph, for the same force was employed in obtaining each series.

The product of the number of mass units possessed by a body and the speed at which that body is moving is termed momentum.

The number of mass units being $\frac{W}{g}$ then the product $\frac{W}{g} V$ is momentum. The slope of the graph (Fig. 5), viz., $\frac{WV}{gT}$, is the increase per sec. of the product $\frac{W}{g} V$, that is, the increase per sec. of the momentum imparted to the trolly. The accelerating force F being equal to the slope of the graph, it follows that F equals the rate of change of momentum of the moving trolly.

It has thus been demonstrated that the force accelerating a body is *equal* to the rate of change of momentum of the body subjected to the action of the force. According to modern views,

the word *force* cannot express anything more than the rate in question ; that is to say, "*force" means rate of change of momentum.*

When one speaks of a "push" or a "pull," these terms may be regarded as implying examples of forces (commonly, the exertion of a muscular effort is understood), but they certainly do not express the meaning of the word "force" in terms of other conceptions. The statement at the commencement of "Applied Mechanics—First Year," that force means a push or a pull, was therefore purely provisional.

When the unit of force is a poundal the quantity of matter which weighs 1 lb. possesses one mass unit. When force is expressed in poundals the number of units of momentum of a piece of substance weighing W lb. and moving at a speed of V ft. per sec. is WV.

In the metric system the unit of force is a dyne, and a quantity of matter which weighs one gram possesses a unit of mass. A portion of matter moving at a speed of 1 cm. per sec., and weighing 1 gram, possesses a unit of momentum.

When the speed of an object in a given direction is denoted, "plus" the momentum of that object is plus also. Since speed in the opposite direction will then be "minus," the momentum of the object will be minus also. The speed of a body along its path at a particular instant may be resolved into component velocities, and in any particular case its mass must remain constant. Hence a quantity of momentum may also be resolved into component effects in different directions.

When we are considering momentum in a particular direction we say that

$$\text{momentum} = \text{mass} \times \text{velocity}.$$

This equation is easily remembered, and may be regarded as a summary of the discussion, if by velocity the actual speed of the body is understood.

Kinetic Energy—On p. 3 is a table giving the lengths of the intercepts formed by a curve traced by a vibrating spring upon a strip of paper attached to a trolly (Fig. 1). The length of each intercept is equal to the distance moved by the trolly in a period of 0·2 sec. when it is accelerated by the weight of the load F shown in Fig. 1. The lengths of the intercepts enable us to find the speed of the trolly at particular instants ; the speed so calculated is that which occurs when the tracing pencil is passing the mid-interval point of the intercept. For the intercept C the mid-interval point is marked M in Fig. 2.

The point O in the same figure being the zero of the curve, the length OM is the distance travelled by the trolly when the tracing pencil is at M. Thus, by measuring all such distances as OM and finding the speed of the trolly when the tracing pencil is at the points in other intercepts which correspond to M (see p. 2), we obtain

numbers which, when plotted, result in a graph resembling A or B in Fig. 6.

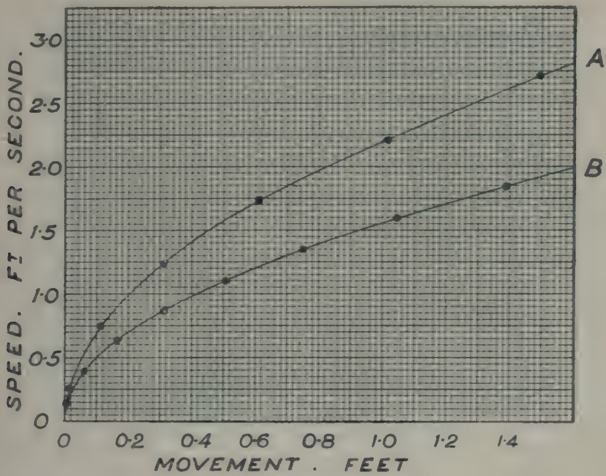

FIG. 6.

When the weight of the trolly, together with the weight of the load F, amounted to 3·925 lb., that of the load F being 0·3 lb., measurements from the curve formed by the vibrator yielded the following results :—

Time of movement (secs.)	0·1	0·3	0·5	0·7	0·9	1·1
Movement of trolly (ft.)	0·0125	0·1125	0·311	0·61	1·017	1·492
Speed of trolly (ft. per sec.)	0·26	0·75	1·235	1·73	2·22	2·71

These numbers were plotted to form the graph A. The graph B was obtained when the total weight of the trolly, together with that of the load F, was 7·85 lb. The amount of the load F was still 0·3 lb. The measurements are as follows :—

Time of movement (secs.) . .	0·1	0·3	0·5	0·7	0·9	1·1	1·3	1·5
Movement of trolly (ft.)	0·0067	0·06	0·163	0·3125	0·5075	0·75	1·044	1·387
Speed of trolly (ft. per sec.). .	0·137	0·39	0·633	0·862	1·1	1·346	1·596	1·846

The time of movement is given merely for reference; in neither case is it used in plotting the graph. Each graph is suggestive of a parabola; if such be the case

$$\text{speed} = \sqrt{k \times \text{movement}},$$

or

$$[\text{speed}]^2 = k \times \text{movement},$$

when k is some constant number. Whether the curve conforms to this equation or not may be decided by finding the square of each number representing a speed and then plotting these squares against the numbers which show the movement of the trolly. The results obtained by carrying out these operations are exhibited in Fig. 7. The numbers used in plotting the graphs are those contained in the first and third lines of each of the tables below.

Movement (ft.) . . .	0·0125	0·1125	0·311	0·61	1·017	1·492
Speed (ft. per sec.) . .	0·26	0·75	1·235	1·73	2·22	2·71
[Speed, ft. per sec.]² . .	0·067	0·562	1·52	2·99	4·92	7·34

The numbers above refer to graph A; for graph B the numbers are

Movement (ft.)	0·0067	0·06	0·163	0·3125	0·5075	0·75	1·044	1·387
Speed (ft. per sec.)	0·137	0·39	0·633	0·862	1·1	1·346	1·596	1·846
[Speed ft. per sec.]²	0·0187	0·152	0·4	0·743	1·21	1·81	2·55	3·41

As each graph is a straight line passing through the origin, it is evident that the equation

$$[\text{speed}]^2 = k \times \text{movement}$$

is a correct interpretation of the results. The constant k is the slope of the graph. For A, the value of k is 4·9; for B, it is 2·46. The significance of these results will be apparent when the following facts are appreciated.

In order to raise a portion of substance weighing W lb. to a height of H ft. above the ground, WH ft.-lb. of work must be done, and consequently WH ft.-lb. of energy must be expended. This energy is recoverable, for at any time the material may be allowed to descend; gravitational force, the weight of the substance, that is, will be exerted through the distance of descent, and hence as much work may be performed during the descent as was required to raise the material. The recoverable energy is called *energy of*

position ; energy which is due to any cause other than the possession of motion by a body is termed *potential energy.* Energy of position is therefore a particular case of potential energy.

Let S ft. denote at any chosen point the descent of the load F in Fig. 1 from its highest position. If at the commencement the load was H ft. above the ground, its initial energy of position was FH ft.-lb., for F represents its weight in lb. ; when it is nearer the ground by S ft. the energy of position is F (H — S) ft.-lb. ; the energy of position has therefore diminished by FS ft.-lb. As the diminution occurs speed is acquired by the load and the trolly ; as no other alteration has taken place we conclude that FS ft.-lb.

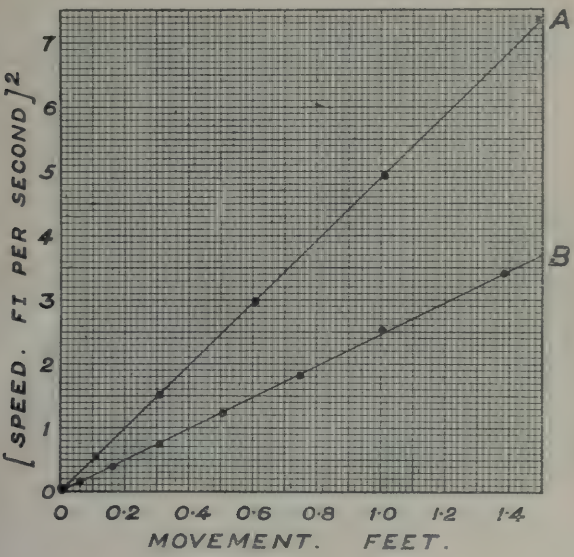

FIG. 7.

of energy have been expended in producing this speed. The energy expended in causing movement is proportional to the amount of movement and—as the graphs in Fig. 7 show—the square of the speed is also proportional to the amount of movement. We conclude, therefore, that the expenditure of energy of position is proportional to the square of the speed acquired by the moving trolly.

That the energy of the moving trolly is also proportional to its mass is a reasonable assumption. Suppose the trolly and the load F to be duplicated. Regarding the two trollys with their falling loads as one piece of material, we shall have twice the amount of mass as at first. As there are now two equal falling loads, there

will be twice as great a force acting as at first, but it should be evident that no change takes place in the amount of movement or the speed attained in any given period of time. It follows that since the expenditure of energy and the mass acted upon are both doubled whilst the speed is unaffected, the energy of a moving body is proportional to its mass as well as to the square of its speed. All this is expressed in the equation

$$C \times \text{energy of a moving body} = \text{mass} \times [\text{speed}]^2.$$

C is a number, of which the value will appear later.

Let E denote a number of units of energy, *m* a number of units of mass, and V a number of units of speed. Restating the equation, we have

$$C \times E = m \times V^2.$$

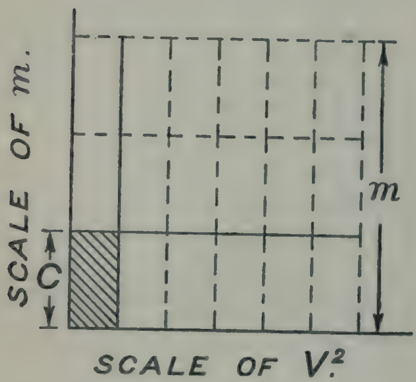

FIG. 8.

The meaning of this is illustrated by Fig. 8.

Let the values of V^2 be represented upon a horizontal scale and the value of *m* upon a vertical scale. The mass of a moving body possessing one unit of energy when V^2 is equal to 1 is represented upon the vertical scale by the length C. Then the small shaded rectangle represents a unit of energy.

If the mass represented by C remains constant, E increases proportionately to V^2, and for each unit added to the value of V^2 a unit of area is added to the bottom row of rectangles. Let the value of V^2 be maintained at unity and *m* be increased. The corresponding amount of energy is now represented by the vertical row of rectangles above the shaded area, each rectangle representing the addition of a quantity of mass amounting to C. For *m* units of mass the number of horizontal rows of rectangles will be $\frac{m}{C}$. The value of E corresponding to any given values of *m* and V^2 will therefore be $\frac{m}{C} \times V^2$, or

$$CE = mV^2$$

as previously stated.

Let the weight of a trolly and that of a load causing acceleration together amount to W lb. F lb. being the weight of the load alone,

the accelerating force is F lb. In T sec. let the distance moved be S ft. and the velocity attained be V ft. per sec. On p. 13 it is shown that

$$F = \frac{W}{g} \times \frac{V}{T}.$$

The acceleration being constant,

$$S = \tfrac{1}{2} VT.$$

From these equations we have

$$F \times S = \frac{WV}{gT} \times \frac{VT}{2} = \frac{WV^2}{2g}.$$

F × S represents the energy exerted by a gravitational pull of F lb. through a distance of S ft. This energy has been wholly applied to the production of a speed V in a body possessing $\frac{W}{g}$ units of mass. Hence the expression $\frac{WV^2}{2g}$ represents the number of ft.-lb. of energy possessed by the body by virtue of its speed. *The energy of a body due to its speed is termed Kinetic Energy.* A common abbreviation is K.E. The equation

$$C \times \text{energy of a moving body} = \text{mass} \times [\text{speed}]^2$$

may now be restated thus :—

$$C \times \frac{WV^2}{2g} = \frac{W}{g} \times V^2,$$

which shows that

$$C = 2.$$

The expression which gives the number of ft.-lb. of kinetic energy of a moving body, viz. $\frac{WV^2}{2g}$, is of great importance, but it is applicable only when the engineers' unit of mass is employed. Instead of representing the number of mass units by $\frac{W}{g}$ we may use the general symbol m; for any system of units, therefore,

$$\text{kinetic energy} = \frac{mV^2}{2}.$$

In the metric system of units a unit of energy is called an erg, and is the amount of energy expended in exerting a force of 1 dyne through a distance of 1 cm.

In the last equation, therefore, kinetic energy is expressed by a number of ergs when m represents grams and V cm. per sec.

If the mass of 1 lb. of substance is regarded as a unit, the corresponding force unit is 1 poundal and the unit of energy is 1 ft.-poundal; this being the energy expended in exerting a force of 1 poundal through a distance of 1 ft. Hence, if W represents lb. and V ft. per sec.,

$$\text{ft.-poundals of kinetic energy} = \frac{WV^2}{2}.$$

In order to test the soundness of our conclusions, refer to the table of experimental results given on p. 16 which applies to the case when the total mass of the moving body amounted to $\frac{7 \cdot 85}{g}$ engineers' units. From the last set of observations in the table referred to we find that the speed is 1·846 ft. per sec. after the trolly has moved 1·387 ft. Therefore

$$\frac{WV^2}{2g} = \frac{7 \cdot 85 \times (1 \cdot 846)^2}{2 \times 32 \cdot 18} = 0 \cdot 416 \text{ ft.-lb.}$$

F being a force of 0·3 lb.,

$$F \times S = 0 \cdot 3 \times 1 \cdot 387 = 0 \cdot 4161 \text{ ft.-lb.}$$

Seeing that the calculations above are based upon measurements obtained by methods by no means refined, the agreement of the two results is remarkably close.

One more point is yet to be dealt with. On p. 16 the slopes of the graphs A and B in Fig. 7 are given as 4·9 and 2·46 respectively. What do these numbers express?

Restate the equation,

$$F \times S = \frac{WV^2}{2g}.$$

Putting $\frac{W}{g}a$ for F we have

$$\frac{W}{g}a \times S = \frac{WV^2}{2g},$$

which gives

$$V^2 = 2aS.$$

Each of the graphs in Fig. 7 has a different value for the acceleration a, but, taking each graph separately, a is a constant. Comparing

the last equation, in which V denotes speed and S denotes movement, with the equation to either graph, viz.,

$$[\text{speed}]^2 = k \times \text{movement},$$

it is plain that k, the slope of the graph, represents twice the acceleration. For the graph marked A in Fig. 7 the acceleration is 2·46 (see p. 16). Hence,

$$2 \times \text{acceleration} = 2 \times 2{\cdot}46 = 4{\cdot}92.$$

This agrees with the slope of the graph, viz., 4·9.

For the case B the acceleration was found to be 1·234. Hence,

$$2 \times \text{acceleration} = 2 \times 1{\cdot}234 = 2{\cdot}468.$$

As the slope of the graph B in Fig. 7 was found to be 2·46 agreement again occurs. "Agreement" does not imply perfect equality, for experimental results are rarely obtained with absolute exactitude.

A bullet, moving at the rate of 1,100 *ft. per sec., passes through a thin plank and comes out with a speed of* 1,000 *ft. per sec.; if it then passes through another plank exactly like the former, with what speed will it come out of the second plank?*

The K.E. expended in piercing the second plank is equal to that expended in piercing the first one. The speed only is changed, and K.E. varies as the square of the speed; hence if the required speed is denoted V,

$$(1{,}100)^2 - (1{,}000)^2 = (1{,}000)^2 - V^2$$

$$V^2 = 2(1{,}000)^2 - (1{,}100)^2 = 790{,}000$$

$$V = \sqrt{790{,}000} = 890 \text{ ft. per sec. (approx.).}$$

A loaded truck weighing 10 *tons and moving along a railway siding at a speed of* 5 *miles per hour is brought to rest by a hydraulic stop-block. If the buffer of the stop-block is forced back to the extent of* 4 *ft., what is the average force exerted upon it?*

Let F lb. be the required average force.

$$\text{K.E. of the truck} = \text{work done upon the stop-block.}$$

$$\frac{WV^2}{2g} \text{ ft.-lb.} = (F \times S) \text{ ft.-lb.}$$

$$\frac{10 \times 2{,}240}{2 \times 32{\cdot}2} \times \left[\frac{5 \times 5{,}280}{60 \times 60}\right]^2 = F \times 4.$$

$$F = 4{,}675 \text{ lb.}$$

Fly-wheels.—Owing to the continual variation of the turning effort exerted upon the crank-shaft of a reciprocating engine, energy is not supplied to the crank-shaft at a uniform rate. In

most cases a uniform flow of energy from the engine is desired; it is in order to approximate to this requirement that fly-wheels are used.

Fig. 9 represents an apparatus by means of which the action of the forces in a reciprocating engine may be imitated. A disk A is free to turn about a central gudgeon-pin P. A pin B represents the crank-pin of the engine. Two rollers C C perform the function of the slipper blocks by constraining a point in the rod BD to move in a vertical line which intersects the axis of the gudgeon-pin.

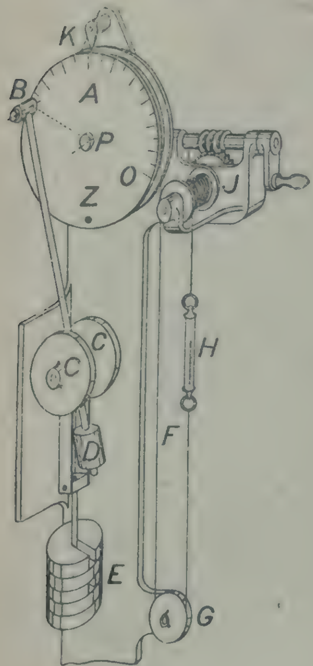

Fig. 9.

At the lower end of the rod a counterpoise is fixed so as to cause the rod to balance about the pin passing through the rollers. From the same pin depends a load pillar carrying a load E. The distance BP is the length of the crank, and the distance from B to the pin passing through the rollers is the length of the connecting rod.

The disk A is made smaller at its middle part so that the centre of a cord F, fixed to and wound around the disk, is at the same distance from the gudgeon-pin as is the crank-pin B. The cord is passed around a grooved pulley G and attached to a spring-balance H. Another cord, fastened to the upper end of the balance, is wound around a drum J which is operated by a worm and worm-wheel.

When the crank-pin is at its zero, position Z, the point O—which is the zero of a scale of degrees engraved on the face of the disk—is opposite a pointer K. When the disk is turned by winding the cord on the drum J, the angle through which the crank moves is indicated by the pointer K upon the degree scale.

The crank-pin is arranged to project beyond the back of the disk A, and after passing through rather more than 180° the projecting part is restrained from moving further by a peg fixed in the upper part of the frame at a point slightly to the right of the vertical centre line. The moving parts are thus prevented from falling over to the right hand.

Turning the disk forward, readings of the spring-balance H were taken at intervals of 10°. The disk was now turned backward and a new set of readings obtained at the same points as before. The second set was not in agreement with the first on account of

frictional resistance. The average of the forward and backward readings at any crank position gives the crank effort. The length of the crank was 0·473 ft., and hence the movement of the crank-pin is found by the following equation :—

$$\text{movement of crank-pin (ft.)} = \frac{\text{crank angle in degrees}}{180^\circ} \times \pi \times 0{\cdot}473$$

Fig. 10 shows the crank effort plotted against the movement of

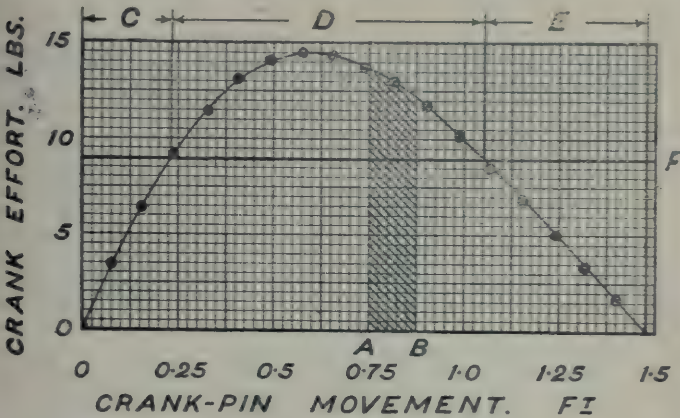

FIG. 10.

the crank-pin, the numbers obtained by the experiment being as follows :—

Crank angle (degrees)	0	10	20	30	40	50	60
Crank-pin movement (ft.)	0	0·0825	0·165	0·2475	0·33	0·412	0·495
Crank effort (lb.)	0	3·437	6·437	9·187	11·437	13·062	14·062

Crank angle (degrees)	70	80	90	100	110	120	130
Crank-pin movement (ft.)	0·577	0·66	0·742	0·824	0·907	0·99	1·072
Crank effort (lb.)	14·5	14·375	13·687	13·0	11·75	10·125	8·5

Crank angle (degrees)	140	150	160	170	180
Crank-pin movement (ft.)	1·154	1·237	1·32	1·403	1·485
Crank effort (lb.)	6·812	5·0	3·312	1·687	0

The load lifted, including the weight of the rods, etc., was 13·96 lb., and this force may be regarded as equivalent to the pressure upon the piston of an engine. The product of the average crank effort and the distance through which it moves expresses the energy imparted to the crank-shaft during the movement. In Fig. 10 the shaded area shows the energy imparted to the crank-pin as it moved between the points A and B in its path indicated on the horizontal scale.

The average crank effort during the total movement from the first to the second dead centre is found by dividing the area under the graph into narrow strips, adding together the readings on the vertical scale indicated by the centre lines of the strips, and then dividing the sum by the number of strips. The results of this operation with respect to Fig. 10, when the diagram was divided into eighteen strips, were as follows :—

$$\begin{aligned}
\text{Sum of crank efforts} &= 161{\cdot}02 \text{ lb.} \\
\text{Average crank effort} &= 8{\cdot}945 \text{ lb.} \\
\text{Length of crank} &= 0{\cdot}473 \text{ ft.} \\
\text{Total movement of crank-pin} &= \pi \times 0{\cdot}473 \text{ ft.} = 1{\cdot}485 \text{ ft.} \\
\text{Energy represented by the diagram} &= 1{\cdot}485 \text{ ft.} \times 8{\cdot}945 \text{ lb.} \\
&= 13{\cdot}28 \text{ ft.-lb.}
\end{aligned}$$

Energy was supplied to the crank disk by pulling the cord around it and this energy was expended in lifting a load of 13·96 lb. through a height of $2 \times 0{\cdot}473$ ft. or 0·946 ft.

Hence,

$$\text{work done} = 0{\cdot}946 \text{ ft.} \times 13{\cdot}96 \text{ lb.} = 13{\cdot}28 \text{ ft.-lb.}$$

The result agrees with the measurements obtained from the graph.

In an actual engine the operations are reversed, the energy being applied to the piston and transmitted to the crank-shaft ; but the varying rate at which the crank-shaft would receive energy if a uniform pressure were acting on the piston is shown exactly by the graph.

The horizontal graph F represents the uniform outward flow of energy. This graph passes through the point on the vertical scale which corresponds to the average crank effort. It is evident that during the movement of the crank-pin between the points A and B, the energy exerted upon the crank-pin and represented by the shaded area is a greater quantity than the amount of energy carried away from the engine during the same interval. The excess energy is shown by the portion of the shaded area above the line F.

The movement of the crank-pin during one piston stroke is divided up into three parts, C, D, and E. During the first part, the work done on the piston represents a smaller quantity of energy supplied to the crank-shaft than flows from it ; during the second

part an excess of energy is supplied; and during the third interval there is again a deficiency.

It is to act as a compensator, absorbing excess energy at one time and making good a deficiency at another, that a fly-wheel is required. When energy is absorbed the wheel increases its speed; when the wheel parts with energy its speed is correspondingly lowered. The limits of the speed variation depend upon the mass and the average speed of the wheel rim. The greater the mass or the average speed of the rim, the less will be the fluctuation of speed of the wheel.

The load on an engine being quite steady, the speed of the fly-wheel is found to fluctuate between 100 *and* 105 *revolutions per min. The mean diameter of the wheel is* 8 *ft. and the weight of the rim is* 3,000 *lb. What quantity of energy is acquired by the rim during the increase from the lower to the higher speed?*

Linear speed of rim at 100 revs. per min. (ft. per sec.) $= \frac{100}{60} \times \pi 8 = 13 \cdot \dot{3} \pi$

Linear speed of rim at 105 revs. per min. (ft. per sec.) $= \frac{105}{60} \times \pi 8 = 14 \pi.$

The K.E. of the rim at any instant is $\frac{WV^2}{2g}$ ft.-lb. when W is the weight in lb. and V the speed in ft. per sec. If V_1 represents the higher and V_2 the lower speed,

K.E. acquired during change of speed $= \frac{W}{2g}(V_1{}^2 - V_2{}^2)$

$$= \frac{3{,}000}{2 \times 32 \cdot 2} \times (14\pi)^2 - (13 \cdot \dot{3}\pi)^2$$

$$\doteq 8{,}850 \text{ ft.-lb.}$$

No account has been taken of the influence of the arms and the boss of the wheel; more energy has been absorbed by the *wheel* than is shown by the result obtained, because the arms and boss share the function of the rim. As, however, the question was limited to the action of the rim, the result is not to be regarded as incorrect.

When an engine is provided with two cranks, one being set at a suitable angle to the other, the excess energy developed on one piston is transmitted to the crank-shaft at the time when a deficient supply is received from the other. In such a case, the excess of energy per revolution to be absorbed by the fly-wheel is a much smaller fraction of the amount developed per revolution, and consequently a smaller fly-wheel is required to maintain given limits of speed fluctuation.

Impulse.—The principles of momentum are of importance in problems concerning the design of turbines and centrifugal pumps. Experiments which throw light upon the subject may be carried out with the aid of a modification of the apparatus shown in Fig. 1. Two trollys, A and B in Fig. 11, and two vibrating springs C and D are now necessary. The board E is about four times as long as

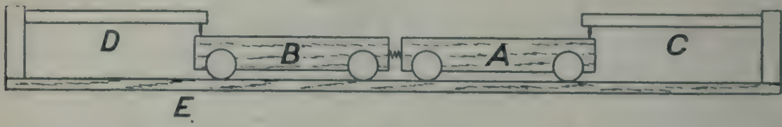

FIG. 11.

one of the trollys; the cord and pulley shown in Fig. 1 are no longer needed. The board cannot be adjusted to compensate for all the friction effects due to two objects moving in opposite directions.

Fig. 12 illustrates a device for connecting the two trollys. Upon the end of each a circular brass plate is screwed, one of them being marked A and the other B. A screw C fitting a tapped hole in a boss extending from the plate A is used to clamp a circular rod D.

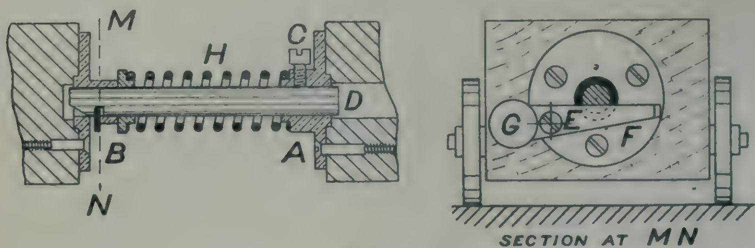

FIG. 12.

This rod passes quite freely through a hole in the centre of a boss on the other plate B. The left-hand end of the rod is bevelled as shown in the figure.

Upon the plate B is fixed a gudgeon-pin E, about which a lever F is free to turn. A piece of metal G attached to the left-hand arm of the lever is heavy enough to overbalance the weight of the right-hand arm of the lever. The lever enters a slot in the boss and engages in a notch near the end of the rod D.

A short, stiff helical spring H grips the right-hand boss firmly, and is provided at the left-hand end with a brass washer which slides easily upon the rod. By means of the screw C the rod is so adjusted that when the left-hand end is pushed through the plate B

far enough for the notch to engage with the lever, the spring is in a state of compression.

Upon a strip of paper pinned to the upper surface of each trolly a centre line is traced by rolling the trollys beneath the pencils on the ends of the vibrators, these, of course, remaining at rest.

Let the trollys now be placed in the positions shown in Fig. 11. Set the vibrators in motion and knock downward, with the handle of a bradawl, say, the right-hand end of the lever. The lever being depressed, the rod E is unlocked and the compressed spring causes the trollys to move, one to the right, the other to the left. Curves are traced by the vibrator pencils ; Fig. 13 represents one obtained in this manner.

The point O corresponds to the position of the trolly at the commencement of the motion. Four intercepts, P, Q, R, and S,

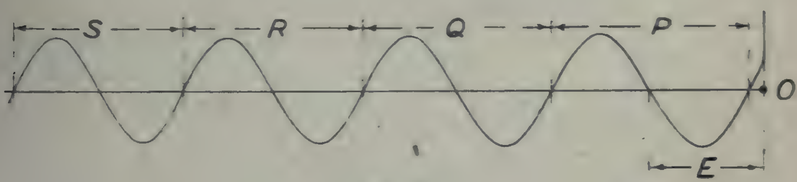

FIG. 13.

were measured. As each intercept was formed in 0·2 second, the average speed of formation is found by multiplying its length by 5. The length E represents the distance moved by the trolly when the average speed of formation of the intercept P coincided with the actual speed of the trolly.

Plotting the speed of the trolly vertically against the distance moved, the two graphs in Fig. 14 are obtained. Because only a few intercepts of the curve are used the graphs are practically straight. The inclination of each graph shows that the speed drops as the movement continues, an effect which is attributable to frictional resistance. The initial speed is indicated by the length of the intercept formed on the vertical axis.

The assumption is made that the initial speed was obtained instantaneously, whereas it was actually acquired during the very short period occupied by the extension of the spring ; no appreciable error being caused thereby, the small movement taking place during this interval is disregarded.

In carrying out these operations results were obtained as follows :—

Weight of trolly A = 5·32 lb.
Weight of trolly B = 7·59 lb.

Intercept		P	Q	R	S
Length (in.)	for A	4·47	4·34	4·26	4·10
	for B	3·10	2·96	2·82	2·69
Speed (in. per sec.)	for A	22·35	21·70	21·30	20·80
	for B	15·50	14·80	14·10	13·45
Movement (in.)	for A	2·82	7·28	11·6	15·82
	for B	1·77	4·79	7·69	10·43

Fig. 14 shows speed plotted against movement. When the graph was drawn to a much more open scale than in the figure, the

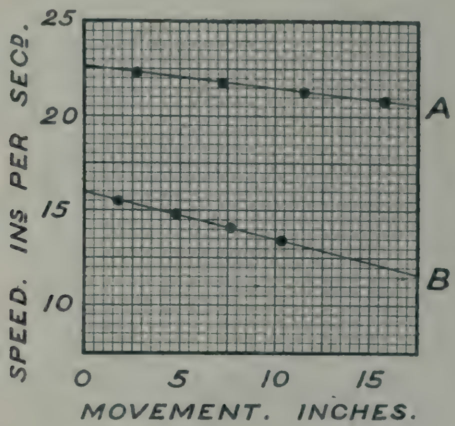

Fig. 14.

initial speed of A was found to be 22·6 in. or 1·883 ft. per sec. Similarly, the initial speed of B was found to be 15·9 in. or 1·325 ft. per sec. We now have

$$\text{momentum of A} = \frac{W}{g} V = \frac{5{\cdot}32}{32{\cdot}2} \times 1{\cdot}883 = 0{\cdot}311$$

$$\text{momentum of B} = \frac{W}{g} V = \frac{7{\cdot}59}{32{\cdot}2} \times 1{\cdot}325 = 0{\cdot}312.$$

The resulting numbers express momentum in engineers' units, and they indicate that equal amounts of momentum have been generated in the two trollys. The reason may now be enquired into.

Referring to p. 13 we find that if a force of F lb. acts for T

sec. upon a body possessing a mass of $\frac{W}{g}$ mass units, a speed of V ft. per sec. is imparted to the body, and

$$F = \frac{W}{g} \times \frac{V}{T}.$$

Hence,

$$FT = \frac{W}{g} V.$$

The product of the numbers representing force and time of action, viz., FT, is the measure of what is termed an *impulse.*

The number expressing the impulse equals the one which expresses the momentum generated by the impulse, or

impulse = momentum generated.

Denoting mass units by m, then for any system of units

$$FT = mV.$$

In this form it may be most conveniently impressed upon the memory.

In applying the result to the last experiment it must be realised that when a steel spring is in compression, equal forces are exerted in each direction along a straight line, and as the compressive force becomes less when the spring extends, equal reductions of force, occurring simultaneously, take place in each direction. This behaviour of the spring is an example of the principle known as Newton's Second Law of Motion, and stated thus,

Action and reaction are mutual and opposite.

When a gun is fired, the pressure of the gas generated by the explosion of the charge acts upon the projectile and the gun in the same way as the spring acts upon the two trollys in the experiment just described. Consequently the projectile and the gun acquire equal amounts of momentum due to the impulse exerted by the expanding gas.

A gun weighs 14 *tons. If a projectile weighing* 250 *lb. leaves the muzzle of the gun at a speed of* 2,000 *ft. per sec., at what speed will the gun recoil?*

It is assumed that the recoil is unchecked. Let V ft. per sec. be the required speed of recoil.

Momentum of projectile = momentum of gun.

$$\frac{250}{g} \times 2{,}000 = \frac{14 \times 2{,}240}{g} \times V$$

$$500{,}000 = 31{,}360\ V.$$

$$V = \frac{500{,}000}{31{,}360} \text{ ft. per sec.} = \begin{array}{c}\text{16 ft. per sec.}\\ \text{(nearly).}\end{array}$$

Collision.—A further modification of the apparatus is now required. In Fig. 15 the end of one trolly is shown provided with a sharp steel spike A screwed into a plate B. A short tube C is used to protect the spike when not in use. A plug of soft wood D is fixed to the end of the other trolly.

Removing the tube C, place the trolly B in the position shown

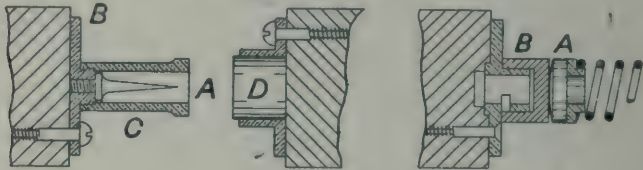

FIG. 15.

in Fig. 16, after obtaining a centre line with the vibrator at rest. Obtain a similar centre line on the other trolly, and after setting both vibrators in motion, impel the trolly A toward the other by means of a sharp push.

When the moving trolly reaches the stationary one, the spike enters the plug D and the two trollys move forward at a common speed. From the intercepts of the curves traced by the vibrators, the speed of the moving trolly just before collision and the common speed of the two just after collision may be determined by the

FIG. 16.

method given on p. 27. In this manner the following results were obtained :—

Weight of moving trolly A	$= W_1 = 7{\cdot}71$ lb.
Weight of stationary trolly B	$= W_2 = 4{\cdot}51$ lb.
Speed just before collision of trolly A	$= V_1 = 2{\cdot}71$ ft. per sec.
Speed of both trollys after collision	$= V_2 = 1{\cdot}71$ ft. per sec.
Momentum of trolly A before collision	$= \dfrac{W_1V_1}{g} = \dfrac{7{\cdot}71 \times 2{\cdot}71}{32{\cdot}2} = 0{\cdot}65$
Momentum of two trollys together after collision	$= \dfrac{W_1 + W_2}{g} V_2$
	$= \dfrac{(7{\cdot}71 + 4{\cdot}51)\ 1{\cdot}71}{32{\cdot}2} = 0{\cdot}65$

The resulting numbers show that the momentum of the two moving

trollys after collision is the same as the momentum of the one moving trolly before collision occurred.

Since a stationary trolly possesses no momentum we have

total momentum before collision = total momentum after collision.

It is not difficult to explain this matter. When the spike entered the plug, a resisting force was encountered due to the inertia of the trolly B, which at any instant was equal and opposite to the force exerted by the trolly A. Let F lb. represent the average amount of either of these equal forces and T seconds the time over which the action extended. Then if Q units of momentum result from the impulse FT, Q units of momentum will be imparted to the trolly B in the direction of the movement, and Q units of momentum will be imparted to the trolly A in the opposite direction. The latter effect is equivalent to depriving the trolly A of Q units of momentum. To the moving trolly we have therefore added — Q units; to the stationary trolly, + Q units. The total initial momentum therefore suffers no change, both trollys being considered.

Since the force used to drive the spike into the plug was exerted through a distance, some of the initial K.E. of the trolly B must have been expended in performing work. The loss of energy is calculated by subtracting the final K.E. from the initial K.E.

$$\text{Initial K.E.} = \frac{W_1V_1^2}{2g} = \frac{7{\cdot}71 \times (2{\cdot}71)^2}{2 \times 32{\cdot}2} = 0{\cdot}88 \text{ ft.-lb.}$$

$$\text{Final K.E.} = \frac{(W_1 + W_2)V_2^2}{2g} = \frac{12{\cdot}22 \times (1{\cdot}71)^2}{2 \times 32{\cdot}2} = 0{\cdot}555 \text{ ft.-lb.}$$

$$\text{Loss of K.E.} = \frac{W_1V_1^2 - (W_1 + W_2)V_2^2}{2g} = 0{\cdot}88 - 0{\cdot}555 = 0{\cdot}325 \text{ ft.-lb.}$$

The equation

initial momentum = final momentum

applies equally well when a transfer of momentum takes place between fluid substances. When "feed-water" is supplied to a boiler by means of an injector—which is the name of an appliance used for the purpose stated—a jet of steam issuing at a high speed from the boiler is condensed by means of cold water. A quantity of steam possesses mass and the jet therefore possesses momentum. When the steam is condensed, part of the momentum is transferred to the condensing water.

If the "blow-off cock" of the boiler were opened whilst the working pressure existed in the boiler, the speed of the resulting stream of water would be very much less than that of the steam jet. The momentum of such a stream of water would be due to an impulse caused by the pressure within the boiler. If another stream of water were forced up the blow-off cock pipe at a speed exceeding that of the issuing water, the latter stream would possess

less momentum per lb. than the former, and the result would be that when the two streams met they would join together and enter the boiler at a speed very much less than the initial speed of either.

The injector acts somewhat in this manner, for the mixture of condensed steam and condensing water has sufficient momentum to open a valve against the pressure of the steam and force the water into the boiler.

The pressure in a boiler being such that a jet of steam issuing from it has a speed of 2,000 *ft. per sec., whilst a jet of water issues at a speed of* 75 *ft. per sec., how many lb. of steam are required to enable an injector to supply* 100 *lb. of water to the boiler?*

It is assumed that the feed-water possesses no momentum when supplied to the injector. Let W lb. represent the required quantity of steam. Then assuming that it is sufficient to impart as much momentum to the feed as the same quantity of water would acquire if it were forced out of the boiler,

$$\text{initial momentum} = \text{final momentum}$$

$$\frac{W}{g} \times 2{,}000 = \frac{W + 100}{g} \times 75$$

$$2{,}000\ W = 75W + 7{,}500$$

$$1{,}925\ W = 7{,}500$$

$$W = \frac{7{,}500}{1{,}925} = 3{\cdot}9 \text{ lb. (approx.).}$$

In actual fact, more than double this quantity of steam would, for various reasons, be necessary.

Let the apparatus shown in Fig. 11 be altered in the following manner. Removing the rod D (Fig. 12) from the trolly, insert a steel plug with a rounded face (A, Fig. 15) in the washer at the left-hand end of the spring. From the other trolly detach, or push aside, the lever F, and place a steel cap B upon the boss projecting from that trolly. Both the plug and the cap must fit tightly.

If the trolly B is allowed to remain at rest as before, whilst A is impelled toward it, then all the energy expended in compressing the spring during the first part of the collision is restored during the second part, and each trolly has an independent motion after collision. The weights of the trollys being the same as in the last experiment, the following numbers resulted when the experiment was performed :—

Speed of trolly A, before collision $= V_1 = 2{\cdot}72$ ft. per sec.
Speed of trolly B, after collision $= V_2 = 3{\cdot}4$ ft. per sec.
Speed of trolly A, after collision $= V_3 = 0{\cdot}71$ ft. per sec.

All the movements occurred in one direction.

If W_1 is the weight of the trolly A, and W_2 is that of the trolly B, we have

$$\text{initial momentum} = \frac{W_1}{g} V_1 = \frac{7{\cdot}71 \times 2{\cdot}72}{32{\cdot}2} = 0{\cdot}652$$

$$\text{final momentum} = \frac{W_1}{g} V_3 + \frac{W_2}{g} V_2 = \frac{(7{\cdot}71 \times 0{\cdot}71) + (4{\cdot}51 \times 3{\cdot}4)}{32{\cdot}2} = 0{\cdot}647.$$

Reasoning in the same way as before, we again come to the conclusion that the total momentum before collision is the same as after collision, for a comparison between the initial and final momenta reveals a discrepancy of less than 1 per cent.

EXAMPLES.

The value of g is to be taken as 32·2.

1. A piece of metal, weighing 60 lb., is mounted on a frictionless trolly which runs along a horizontal path. What force in pounds weight will, if applied horizontally in the direction of the motion, produce an acceleration of 12 ft. per sec. per second?
2. The weights of two bodies (A and B) are 7 lb. and 12 lb. respectively. If it were possible to transfer A to a place where a falling body acquires in one second a speed of 32 ft. a sec. and B to a place where a falling body acquires in one sec. a speed of 28 ft. a sec., what would then be the ratio of the weight of A to the weight of B?
3. A locomotive draws a train weighing 400 tons. What pull in the draw-bar (lb.) ~~is~~ necessary to impart an acceleration of 2 ft. per sec. per second to the train?
4. The weight of a number of people, standing on the floor of a lift, amounted to 6,000 lb. At a particular moment in the descent of the lift the pressure upon the floor was 5,400 lb. At what rate, in ft. per sec. per second, was the speed of descent being increased when this occurred?
5. A load weighing 25 lb. is suspended from a spring-balance fixed in the cage of a lift. What will the balance indicate when the cage is lowered with an acceleration of 6 ft. per sec. per second?
6. A force of 40 lb. acts on an object weighing 2 tons; find the acceleration produced. How far will the object be displaced in 5 secs., starting from rest?

7. A body having a mass equal to that of 8 lb., moves in a straight line under the action of a force which, in three sec., increases its speed by 9 ft. a sec.; find the ratio of the force to the weight of the body. Also find the number of poundals in the force.
8. The rim of a fly-wheel weighs 5 cwt. and is 4 ft. in diameter. The wheel is keyed on a shaft to which is attached a crank 6 in. long. The wheel is accelerated from a state of rest by a pressure exerted on the crank-pin, and after making 15 turns rotates at a speed of 95 revs. per min. Find the mean force exerted on the crank-pin in a direction tangential to its circle of revolution.
9. A loaded cage which moves up a pit-shaft weighs 1·125 tons. A loaded trolly, weighing 3 tons, is attached to the cage by a rope passing over a pulley at the pit-head. The trolly is allowed to run down an incline of 30°, the rope attached to it maintaining the same angle of slope. How high will the cage move from a state of rest in 2 sec.?
10. The loaded die of a stamping press weighs 70 lb. The die is allowed to fall under the action of gravity. If 0·9 sec. is the time of falling, what amount of K.E. does the die acquire?
11. A train of 100 tons is going on the level at a speed of 30 miles per hour, when by applying the brakes it is brought to rest in 600 ft.; calculate the average total resistance of the brakes.
12. A loaded trolly, which weighs 1 cwt., moves along a pair of horizontal rails at a speed of 25 ft. per sec., and is brought to rest against a buffer containing a helical spring; a force of 320 lb. is required to compress the spring to the extent of 1 in. How much is the spring compressed in bringing the trolly to rest?
13. A 3-ton cage descending a shaft at a speed of 9 yds. per sec. is brought to rest by a retarding force exerted along the rope in a space of 18 ft. What is the tension in the rope whilst the stoppage is taking place?
14. A piece of metal falling vertically downward at a speed of 10 ft. per sec. and weighing 200 lb. comes in contact with a spring which it compresses to the extent of 1 ft. and then comes to rest. What is the rate of compression of the spring in lb. per ft. of compression?
15. A garden syringe is fitted with a tapering nozzle. The sectional area of the barrel is 1·5 sq. in. and the length of the plunger stroke is 1 ft. When water is ejected 0·4 of the energy is lost on account of frictional resistance. At what speed does the jet issue if a steady pressure of 20 lb. is maintained upon the plunger?
16. The rim of a fly-wheel weighs 2·5 tons and rotates with a tangential speed of 40 ft. per sec. If the speed is reduced until the energy possessed by the rim is 10,000 ft.-lb. less, what will the speed then be?

17. A piece of material which weighs 120 lb. has an initial speed of 200 ft. per sec. A retarding force of 60 poundals being exerted upon it, how far will this object move whilst its speed is reduced by 100 ft. per sec.?
18. A speed of 60 miles per hour is attained in 10 min. by a train which weighs, exclusive of the locomotive, 150 tons. For every ton of weight a pull of 10 lb. is needed to overcome frictional resistance. What is the total force exerted upon the draw-bar of the engine?
19. Refer to Fig. 1. If F represents a 1 lb. weight, which, after descending for 2·2 sec., imparts to the trolly A a speed of 3·6 ft. per sec., what must be the weight of the trolly?
20. A loaded cage, weighing 3 tons, descends the shaft of a mine at a speed of 9 yds. per sec.; by the application of the brake it is brought to rest in a period of 4 sec. What is the tension in the rope whilst the stoppage is taking place?
21. A gun weighing 5 tons fires a shot weighing 30 lb. If the shot leaves the muzzle at a speed of 1,000 ft. per sec. and the recoil of the gun is unchecked, what is the speed of recoil?
22. What number of dynes acting upon a body possessing a mass of 20 grams for a period of 4 sec. will impart to the body a speed of 50 cm. per sec.?
23. The moving parts of a steam hammer weigh half a ton, and fall freely from a height of 6·25 ft. If the force of the blow endures for 0·05 sec., what is its average amount in lb.?
24. The weight of a train is 190 tons; its speed is 50 miles per hr. If by putting on the brakes the train is brought to a standstill in 15 sec., what is the mean resistance?
25. A trolly A is impelled toward a stationary trolly B at a speed of 12 ft. per sec. An automatic coupling prevents them separating after contact takes place. The two then move forward at a speed of 3·6 ft. per sec. Assuming that A weighs 16 lb., what will B weigh?
26. A trolly A weighs 3·8 lb. and is impelled to the left at a speed of 16 ft. per sec. A trolly B weighing 2·2 lb. is impelled to the right at a speed of 14 ft. per sec. The trollys meet and remain in contact. At what speed and in which direction do the two then move?
27. A trolly A weighs 10·5 lb.; a trolly B weighs 8 lb. B remains at rest whilst A is impelled toward it. After impact has occurred, the two move forward together at a speed of 12·6 ft. per sec. What was the speed of A at the moment of impact?
28. A trolly A, weighing 4·86 lb., stands at rest on a level surface. A trolly B, weighing 9·6 lb., is fitted with a spring buffer on its leading end. If B is set in motion and strikes A, no energy is lost, owing to the perfect elasticity of the spring. Suppose that B is impelled toward the left at a speed of 40 ft. per sec.;

after striking A, will the trolly maintain the same direction of motion, and if so, at what speed? If not, at what speed will it return? Also, find the speed imparted to trolly A.

29. A shell weighing 60 lb. and moving at a speed of 500 ft. per sec. bursts into two parts, each of which moves along the original line of motion. The forward part which weighs 35 lb. has its speed doubled by the explosion. At what speed and in which direction does the rear part move?

30. A shell bursting into two pieces, the rear part, which is three times as heavy as the forward part, is brought to a standstill. The forward part continues its motion in the original line of flight of the shell. If the speed of the shell at bursting is 400 ft. per sec., what is the final speed of the forward part?

CHAPTER II

Pressure exerted by Liquids.—Arrange that an object of any shape shall float in a liquid with an appreciable quantity of its bulk above the surface; the float of a ball-cock will answer this purpose. Exert a downward thrust upon the object. So long as it is not totally submerged, the more deeply it is immersed the greater will be the thrust required. It will be evident that an upward force balances the downward one exerted upon the object.

Force relationships where a liquid is concerned may be investigated with the aid of the apparatus represented in Fig. 17.

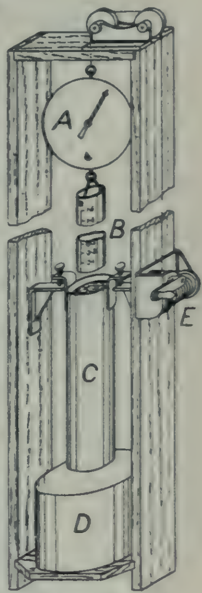

FIG. 17.

A spring-balance A is supported by a cord passing over two grooved pulleys fixed on the top of a frame. To the spring-balance is suspended a tube B about 2 in. in diameter and having its lower end closed by a plug flush with the bottom edge of the tube; its length is from 1·5 ft. to 2 ft. long, and when lowered into water it is heavy enough to sink until the water runs over the top edge; this, however, must not take place when an experiment is in progress. The tube is painted white, and a scale of feet, each foot divided into tenths, is marked on the outside; the zero of the scale is at the level of the bottom edge of the tube.

A vessel C, supported by brackets screwed to the frame, consists of two concentric tubes, the inner one having a diameter about 1 in. greater than that of the tube B; the lower end of the inner tube is plugged. The outer tube has a diameter about $\frac{3}{4}$ in. greater than the inner one, has both ends open, and its upper edge is not more than 0·5 in. above that of the inner tube. A vessel D stands upon a shelf below the vessel C; its upper edge is slightly below the bottom of the vessel C. The cord which supports the tube B is wound upon a drum E operated by a worm and worm-wheel.

Fill the vessel C to the brim with water; any water which overflows into the vessel D should be thrown away. Weigh the vessel D and place it in the position shown in the figure. Before the tube B is allowed to touch the water in the vessel C, find its weight by means of the spring-balance A. Lower B gently until about 0·2 ft. of its length is below the level of the water in C. As the tube B descends, water is displaced from C and flows over the edge of the inner tube; passing between the inner and outer tubes,

this water is caught in the vessel D. As the weight of D is known, it will be easy to find the weight of the water it contains. This being done, the reading of the balance A is observed before any further movement of B takes place. Repeat these operations from ten to twelve times, but allow no water to leave the vessel D until the experiment is concluded.

In an apparatus made as shown in Fig. 17 the tube B had an outside diameter of 1·62 in. and weighed 2·45 lb. The balance readings, depth of immersion, and weight of water caught in the vessel D are given in the following table.

Depth of immersion (ft.)	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0
Reading of balance A (lb.)	2·27	2·12	1·95	1·77	1·58	1·41	1·22	1·03	0·85	0·70
Weight of water in D (lb.)	0·19	0·36	0·54	0·72	0·90	1·08	1·25	1·43	1·61	1·78

As the tube B descends, the reading of the balance A becomes less. Apparently the effect of lowering the tube into the water is to reduce its weight, but this is an impossibility in the case of any

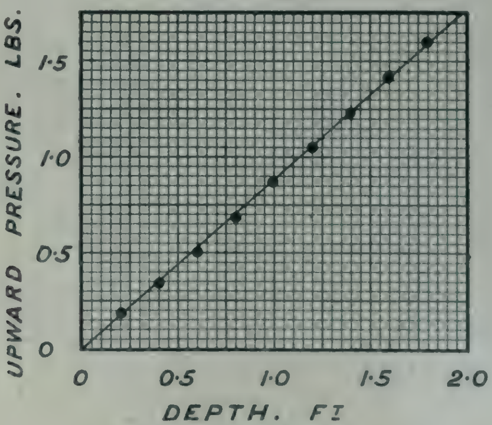

Fig. 18.

object which remains unaltered in other respects. The reduction of the balance reading is due to the upward pressure of the water in the vessel C; for the action which takes place is the same as when a floating body is pushed downward. The table of observations shows that when 0·8 ft. is immersed the downward pull on the balance is 1·77 lb. As the tube B weighs 2·45 lb. the upward pressure due to the water is 2·45 lb. — 1·77 lb. or 0·68 lb. A table

of upward pressures, as given below, is therefore obtained by subtracting each balance reading from the weight of the tube B.

Depth of immersion (ft.) . .	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0
Upward pressure (lb.) . .	0·18	0·33	0·50	0·68	0·87	1·04	1·23	1·42	1·60	1·75

The graph (Fig. 18) of the last two columns of numbers shows that the upward pressure is proportional to the depth of immersion of the tube B. The slope of the graph is 0·89. Compare the upward pressure at each depth of immersion with the weight of water displaced from the vessel C at that depth of immersion. Examination of the results—although there is not perfect agreement—will bear out the following statement :—

When an object is immersed in a liquid, a vertical upward force equal to the weight of the quantity of liquid displaced is exerted upon the object by the liquid surrounding it. The force is termed "buoyancy."

The discrepancy of a few hundredths of a lb. between corresponding readings is due to the fact that the balance was not read accurately, for 0·1 lb. was indicated on the dial by a distance of about 0·1 in.

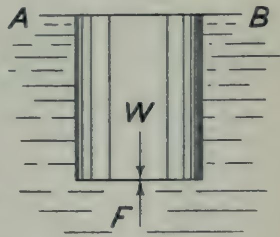

FIG. 19.

In Fig. 19 AB represents the surface of a body of water. Suppose that a cylinder of water, one end being level with the surface, is isolated from the surrounding liquid by an extremely thin casing of a material that when immersed in water will remain without constraint wherever it is placed. The weight W of the cylinder of water is balanced by a force which can only be due to the upward pressure F, equal to W, the buoyancy, that is, of the surrounding water. If all the water were removed from the imaginary cylindrical casing, the force F would still be exerted. If the diameter of this ideal cylinder is the same as that of the tube B in Fig. 17 the latter may be inserted in the space left vacant by the removal of the water. Such a proceeding would be equivalent to what actually occurred in the experiment, and thus the force F is the cause of the indication of the balance A becoming less as the tube descended. By using a series of such tubes as B (Fig. 17), each of which has a different diameter but each diameter being smaller than that of the inner part of the vessel C, further experimental data may be gathered. After the first experiment, nothing is gained by weighing the water caught in the vessel D. For each

tube a graph is plotted showing the upward pressure of the fluid against the depth of immersion, as in Fig. 18. The slope of each graph gives the pressure per ft. depth of immersion for the tube to which it corresponds. The following results were found in this way and are exhibited graphically in Fig. 20.

Pressure per ft. of depth (lb.)	0·326	0·535	0·63	0·89
Sectional area of tube (sq. ft.)	0·00545	0·0084	0·0106	0·0143

The slope of the graph (Fig. 20) is 62·5. The equation to the graph is therefore

$$\text{pressure per ft. of depth (lb.)} = \begin{cases} 62{\cdot}5 \times \text{area of surface acted upon} \\ \text{(sq. ft.)} \end{cases}$$

The surface upon which this pressure acts is the closed lower end of the tube. The suspended tube is free to move sideways but does not do so ; we conclude, therefore, that the horizontal pressures are balanced. Since the pressure for each ft. of depth upon each sq. ft. of horizontal surface is 62·5 lb., and the pressure is proportional to the depth of the surface acted upon, a pressure amounting to 62·5 H lb. is exerted upon each sq. ft. situated at a depth of H ft. If the surface contains A sq. ft. the total pressure is 62·5 AH lb. If the cylinder in Fig. 19 has a sectional area of A sq. ft. and a length of H ft., its volume is A × H cub. ft. As a cub. ft. of water weighs approximately 62·5 lb., the expression 62·5 AH representing the upward pressure of F lb. is equal to the weight W of the cylinder. This agrees with the observed fact that the pressure F is equal to the weight of the water displaced. The matter may be expressed more generally as follows :—

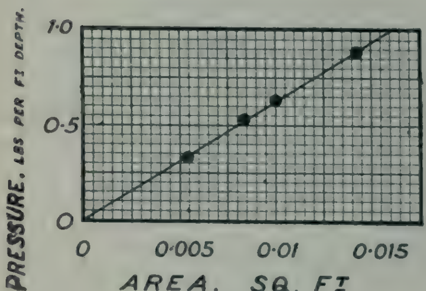

FIG. 20.

When a horizontal flat surface is subjected to the pressure of a liquid, the total pressure upon the surface is proportional to the area of the surface, the height of the free level of the liquid above the surface, and the weight per unit of volume of the liquid.

When the units of measurement are consistent the word "equal" may be substituted for "proportional" in this statement, the equality, of course, existing between two numbers. For engineering purposes the unit of length is usually 1 ft., the unit of area is 1 sq,

ft., the unit of volume is 1 cub. ft. and weight is expressed in lb. when the pressure of a liquid is concerned. Briefly,

$$\left.\begin{array}{l}\text{total pressure in lbs. on a horizontal}\\ \quad\text{surface}\end{array}\right\} = \left\{\begin{array}{l}62{\cdot}5 \times \text{area in sq. ft.}\\ \quad\times \text{ depth in ft.}\end{array}\right.$$

Remember that water was the liquid employed in obtaining the above equation ; consequently we shall not be justified in applying it when some other kind of liquid is used.

The equation says nothing about the *quantity* of water required to produce a given pressure upon a given area. By means of the apparatus represented in Fig. 21 this matter may be investigated.

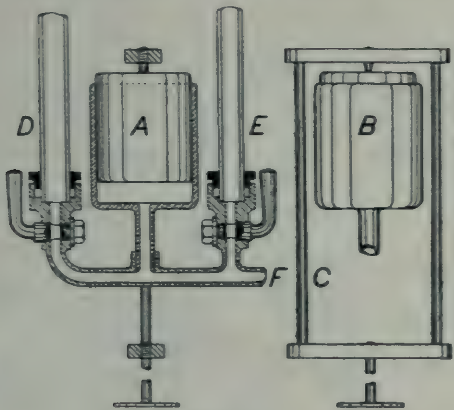

FIG. 21.

A metal cylinder A is made to fit freely, but as perfectly as possible, in a short fixed tube B. The weight of the cylinder A is a force thrusting it downward which may be augmented by a load carried by a stirrup-shaped frame C. Two vertical glass tubes D and E, of unequal diameter, are connected to the inside of the tube B, but the communication between B and D or B and E may be closed by a stop-cock. Through a pipe F a supply of water is forced into the tube B by means of a small pump. By the side of each glass tube is a length scale, the zero of each coinciding with the bottom of the cylinder A when a line engraved around that cylinder is level with the top edge of the tube B. A notch cut in the edge of the tube will allow it to be seen when this occurs.

With both stop-cocks open, force water into the tube B until the line on the cylinder A rises to the top of the tube B. Note the readings of the scales, which are exactly the same. Drain off the water, close one of the cocks, and again pump in water until the cylinder A assumes its previous position. Observe that the scale reading of the single tube is the same as when two were used.

Repeating the experiment with the other tube, the same result follows. Knowing the sectional area and weight of the cylinder A, the last equation stated enables us to calculate the height of the level of the water in the tube. Do this, and if the result agrees with the observed height it may be considered as beyond doubt that the equation takes into account everything which affects the pressure of the water upon the horizontal surface of the cylinder. Some observations made with the apparatus described are as follows :—

Load on cylinder, including its own weight (lb.)	1	2	3	4	5
Height of water in tubes (ft.)	1·34	2·66	4·0	5·3	6·63

The diameter of the cylinder A in this experiment was 1·5 in.

The reader is now in a position to find out for himself whether these results warrant the conclusion that the pressure is not dependent upon the amount of the liquid.

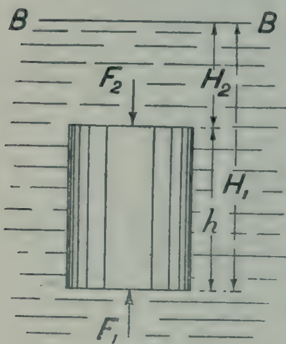

Fig. 22.

Remove the tube B from the apparatus shown in Fig. 17, and replace it by a cylindrical bar of metal supported by a thin, fairly long wire. Lower the bar into the water contained by the tube C until it is entirely submerged. Continue to lower the metal until it has descended nearly to the bottom of C. Observe that after complete submersion takes place no alteration of the balance reading follows, although the upward pressure on the bottom of the piece of metal must increase as the rod descends. The constancy of the balance indication is due to the fact that after submersion takes place there is a downward pressure of the liquid upon the top of the piece of metal which increases at the same rate as the upward pressure upon the bottom.

Fig. 22 illustrates this point. BB represents the water level. The submerged cylinder has a sectional area of A sq. ft. Then, referring to the figure,

$$\text{Force } F_1 = 62{\cdot}5 \times A \times H_1 \text{ (lb.)}$$
$$\text{Force } F_2 = 62{\cdot}5 \times A \times H_2 \text{ (lb.)}$$
$$\text{Resultant force} = F_1 - F_2 = 62{\cdot}5\ A\ (H_1 - H_2) \text{ (lb.)}$$
$$62{\cdot}5\ A \times h \text{ (lb.)}$$

H_1 and H_2 may therefore change to any extent, but as their

difference is always equal to h no alteration of the resultant force will occur to affect the reading of the spring balance.

A tank with sides inclined to the horizontal at an angle θ and filled to the brim with water is represented in Fig. 23. Let our object be to discover what pressure the water exerts on one of the sloping sides. Firstly, in what *direction* is the pressure exerted?

Suppose that a hole T is made in the side. In order to prevent water flowing out, a plate B may be held against the outside. A vertical column of water above the hole is indicated by a shaded area, and the weight of this water is acting downward above the hole. If, however, a vertical upward force F_1 is applied, the plate will slide along the side of the tank. To balance the water pressure a force F_2 must be exerted in a direction normal to the side of the

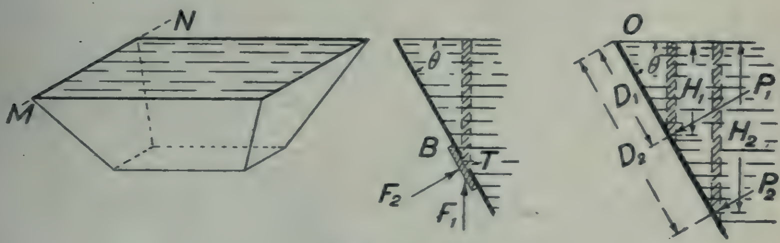

FIG. 23.

tank; the water as well therefore exerts a normal pressure upon the side of the tank. Think of what one sees when a "rose" with a convex outer surface is used with a garden hose. Each jet shoots out in a direction normal to the surface because the motion of the water takes place in the direction of the force impelling it. Hence a force P_1, normal to the sloping side of the tank, is distributed over a narrow horizontal strip, of a length equal to the width of the tank at the place where it is taken. The strip has an area of a_1 sq. ft. and H_1 is the mean height of the water level above it; as the strip is supposed to be exceedingly narrow the mean height does not materially differ from the height of the water surface above either its upper or its lower edge. Hence,

$$P_1 = (62{\cdot}5 \times a_1 \times H_1) \text{ lb.}$$

because $62{\cdot}5\ H_1$ is the pressure in lb. per sq. ft. at the given depth. Similarly, a_2 sq. ft. is the area of another strip at a depth H_2, and

$$P_2 = (62{\cdot}5 \times a_2 \times H_2) \text{ lb.}$$

Refer to the figure; from the properties of similar triangles we deduce that

$$\frac{H_1}{D_1} = \frac{H_2}{D_2} = K \text{ (say).}$$

Therefore,

$$H_1 = KD_1 \text{ and } H_2 = KD_2.$$

Substituting KD_1 for H_1 in the previous equation,

$$P_1 = (62{\cdot}5 \times a_1 \times KD_1) \text{ lb.}$$

If the whole area of the sloping side of the tank be divided up into extremely narrow strips and each be treated in the same manner we shall obtain a vast number of equations resembling the last one. Adding up all the quantities on the right-hand and left-hand sides respectively the result will be

$$\text{total pressure on the surface} = 62{\cdot}5 \text{ K} \begin{pmatrix}\text{sum of all such} \\ \text{terms as } a_1 \text{ D}_1\end{pmatrix} \quad ...(a)$$

The only difficulty, therefore, is to determine the value of the quantity in the brackets.

Suppose the side of the tank to be made of a material which weighs w lb. per sq. ft. If the side of the tank were turned into a horizontal position, the moment about the axis MN of the weight of the strip of plate having the area a_1 would be $w\,a_1\,D_1$; of the lower strip it would be $w\,a_2\,D_2$. If A sq. ft. is the total area of the sloping plate and R is the distance of the centre of gravity of the plate from the axis MN,

$$\text{sum of moments such as } w\,a_1\,D_1 = w\text{AR},$$

and therefore

$$\text{sum of such quantities as } a_1\,D_1 = \text{AR}.$$

Consequently we are justified in putting the alternative value in the equation marked (a), which now becomes

$$\text{total pressure on the surface} = 62{\cdot}5 \text{ KAR}.$$

Let H be the depth of the centre of gravity of the plate forming the side of the tank. Then

$$\text{H} = \text{KR}.$$

Substituting H for KR in the equation above,

$$\text{total pressure on the surface} = 62{\cdot}5 \text{ AH}.$$

The final expression is exactly the same as for a horizontal surface, but H, instead of representing the common depth of all parts of the surface, stands in the present instance for the depth of

the centre of gravity of a plate of uniform thickness which coincides with the area subjected to pressure. It is common to refer to H in this connection as the depth of the centre of gravity of the *area* acted upon. The shape of the surface and its angle of slope do not affect the reasoning employed. Hence the expression applies equally well to a vertical surface.

A sluice-gate is 10 ft. wide and 12 ft. deep. The water on one side is level with the top of the gate. On the other side there is no water. What force is exerted upon the gate by the pressure of the water?

A sluice-gate having a vertical rectangular surface, the depth of the centre of gravity of the area subjected to pressure is $(12 \div 2)$ ft., or 6 ft.

$$\text{Total pressure (lb.)} = 62{\cdot}5 \times (12 \times 10) \text{ sq. ft.} \times 6 \text{ ft.} = 45{,}000 \text{ lb.}$$

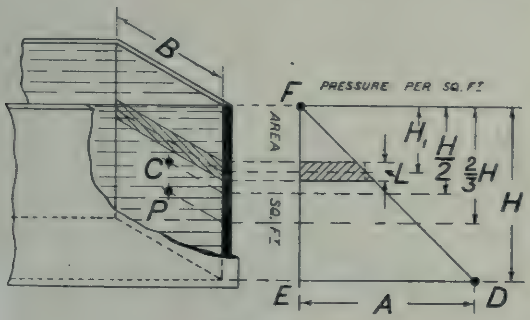

FIG. 24.

The vertical side of a tank is 4 ft. wide. The total pressure exerted upon the side by the water within the tank is 1,000 lb. What is the depth of water?

Let H ft. denote the required depth.

$$\text{Area acted upon} = 4\text{H sq. ft.}$$

$$\left.\begin{array}{l}\text{Depth of centre of gravity of}\\ \quad\text{area acted upon}\end{array}\right\} = \tfrac{1}{2}\text{H ft.}$$

$$\text{Total pressure} = 62{\cdot}5 \times 4\text{H} \times \tfrac{1}{2}\text{H} = 1{,}000 \text{ lb.}$$

$$\therefore \quad \text{H}^2 = \frac{1{,}000}{125} = 8.$$

$$\text{H} = \sqrt{8} \text{ ft. or } 2{\cdot}82 \text{ ft. approx.}$$

Centre of Pressure.—From the fact that the centre of gravity of a surface subjected to the pressure of a liquid is required in finding the amount of pressure on the surface, it may be inferred that the resultant pressure acts through that point. Such is not the case.

Fig. 24 shows the rectangular side of a tank having a width of B ft. and a depth of H ft. At the level of the bottom of the tank set

off a horizontal intercept A representing to a force scale (62·5 H) lb. which is the pressure upon a square foot of area with its centre of gravity H ft. below the surface of the liquid. A scale on the axis FE shows the area of the side of the tank above any given level. FD is a graph showing on the axis of pressure the value of (62·5 H_1) lb., when H_1 ft. is any lesser depth than H. Draw, at any level, two intercepts at a distance L ft. apart, the one as much above as the other is below a chosen depth H_1 ft. The vertical surface between the two levels is BL sq. ft., and the depth of the centre of gravity of this area is H_1 ft. The pressure upon the area is (62·5 BLH_1) lb. which is represented by the shaded part of the diagram between the two horizontal intercepts. The same is true of all other portions of area. A horizontal line on the side of the tank at a depth $\frac{1}{2}$ H ft. contains the centre of gravity of the whole side of the tank. That part of the triangular area which is above the intercept at the depth $\frac{1}{2}$H ft. is only $\frac{1}{3}$ of the area below the same line. Hence, three times as much pressure is exerted on the lower half of the side of the tank as on the upper half. The line of action of the resultant pressure cannot therefore pass through the centre of gravity of the side of the tank.

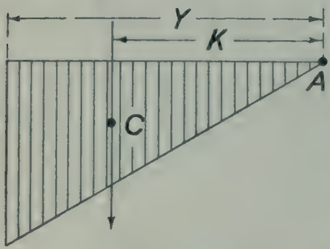

FIG. 25.

Compare the distribution of pressure as shown by the diagram with the manner in which the weight of a triangular sheet of metal is distributed. If the triangle (Fig. 25) is divided into numerous vertical strips of equal width the weight of each is proportional to its length. Compare the weight of one of the strips with the shaded area in Fig. 24. The resultant weight of the triangle (Fig. 25) acts through the point C, which is its centre of gravity. In like manner the resultant of the pressure distributed over the side of the tank passes through the centre of gravity of the triangular area EDF in Fig. 24.

When a triangular sheet of zinc, in which the length Y was 4·95 in., was balanced on a knife-edge the dimension K proved to be 3·29 in.

Hence, $$\frac{K}{Y} = \frac{3·29}{4·95} = 0·665 \text{ (approx.)}.$$

The resulting number is a close approximation to $\frac{2}{3}$. Assume that $\frac{2}{3}$ is the exact value of $\frac{K}{Y}$.

Divide up the triangle as in Fig. 26. If the assumption is correct it will balance on a knife-edge coinciding with XX. The shaded

part is symmetrical about XX and is therefore balanced. Apply the principle of moments to the remaining parts, taking the weight of each triangular part as W. Thus the moment about XX of the triangle on the right is $W \times \frac{4}{3} H$. The moment of the two triangular parts on the left is $2W \times \frac{2}{3} H$. The two results being identical it follows that the complete plate is balanced about XX, and moreover that it cannot be balanced about any other line which is parallel to XX, since such a line could not also contain the centre of gravity. Our assumption that the distance of the centre of gravity of a triangular plate from its base edge is exactly ⅓ of its height is therefore justified.

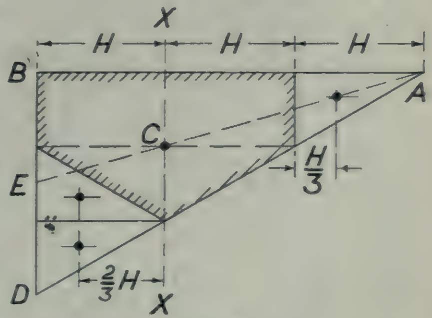

FIG. 26.

Bisect the side BD at E and join E to A. If the plate be divided into extremely narrow strips parallel to BD, as in Fig. 25, each such strip would be bisected by the line EA and each would therefore balance about EA. The whole plate will therefore balance about EA and consequently the point C is the centre of gravity. By applying the principle of similar triangles it is easy to prove that the length CE is ⅓ the length AE. For our present purpose, however, it is sufficient to know that in Fig. 25

$$K = \frac{2}{3} Y.$$

The centre of pressure is the point on a surface through which the resultant fluid pressure upon that surface acts. It coincides with the centre of gravity of the surface only when the surface is horizontal. When the area subjected to pressure is not horizontal, extends to the free surface of the liquid, and is rectangular—two edges being horizontal—the level of the centre of pressure P (Fig. 24) *is at a depth below the free surface of the liquid equal to ⅔ of the depth of the lower edge.*

Specific Gravity.—A piece of steel rod being supported upon a spring-balance, its weight was found to be 5·11 lb. Supporting the rod by a thin wire tied to the balance and then lowering it into water until it was totally submerged, the balance indicated 4·46 lb. Hence,

Weight of water displaced = (5·11—4·46) lb. = 0·65 lb.

The volume of the displaced water is manifestly equal to that of the

metal causing the displacement and may be denoted V units. With its numerical value we are not concerned. Hence,

$$\frac{\text{weight of V units of volume of steel}}{\text{weight of V units of volume of water}} = \frac{5{\cdot}11}{0{\cdot}65} = 7{\cdot}86.$$

The resulting number is termed the specific gravity of the steel.

The specific gravity of a substance is the ratio of the weight of any given volume of that substance to the weight of an equal volume of water.

The same piece of metal was now submerged in paraffin oil, with the result that the balance indicated 4·58 lb.
Hence,

$$\text{weight of paraffin displaced} = (5{\cdot}11 - 4{\cdot}58)\text{ lb.} = 0{\cdot}53\text{ lb.}$$

Consequently,

$$\frac{\text{weight of V units of volume of paraffin}}{\text{weight of V units of volume of water}} = \frac{0{\cdot}53}{0{\cdot}65} = 0{\cdot}815.$$

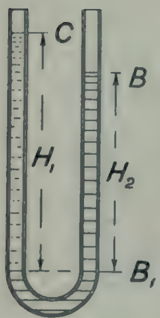

FIG. 27.

The number 0·815 therefore expresses the specific gravity of the paraffin.

A piece of mahogany of which the weight was 0·43 lb. was forced down into the water contained by the vessel C (Fig. 17). The water displaced was found to weigh 0·605 lb. In order that complete submersion should be assured a long thin steel pin with a large head was driven into the wood. The head of the pin was kept above the water level and was used as a handle. The very slight displacement of water due to the stem of the pin was disregarded. In accordance with the observations,

$$\text{specific gravity of mahogany} = \frac{0{\cdot}43\text{ lb.}}{0{\cdot}605\text{ lb.}} = 0{\cdot}71.$$

Refer now to the equation :—

$$\left.\begin{array}{l}\text{Total pressure of a liquid on a}\\ \quad\text{horizontal surface (lb.)}\end{array}\right\} = 62{\cdot}5 \times \text{A sq. ft.} \times \text{H ft.}$$

Since 62·5 represents the weight in lb. of a cub. ft. of water, it is reasonable to suppose that the specific gravity of a liquid will affect its pressure on a submerged surface. That is to say, if water is replaced by some other liquid the number 62·5 will alter in the equation given.

Fig. 27 represents a bent glass tube. When water slightly tinted with a dye was poured into it, the level of the liquid was the same in each of the vertical parts of the tube. Paraffin oil was now poured gently into the left-hand part of the tube, and the water level then rose to B on the right-hand and sank to B_1 on the left.

The level of the paraffin was at C. The heights H_1 and H_2 were now measured, giving the results

$$H_1 = 2{\cdot}315 \text{ ft. and } H_2 = 1{\cdot}88 \text{ ft.}$$

Consider either the lowest film of oil or the highest film of water at the level B, on the left-hand side of the tube. Such a film is in equilibrium, and hence,

$$\left.\begin{array}{l}\text{downward pressure due to } H_1 \text{ ft.}\\ \quad\text{of paraffin}\end{array}\right\} = \left\{\begin{array}{l}\text{upward pressure due to } H_2 \text{ ft.}\\ \quad\text{of water.}\end{array}\right.$$

Below the level B_1, the weight of water in one limb of the tube balances that of the water in the other. Let A sq. ft. be the area of the film. Let W_1 lb. represent the weight per cub. ft. of the paraffin. The last equation may then be expressed as follows :—

$$W_1 \times A \times H_1 = 62{\cdot}5 \times A \times H_2.$$

Hence,

$$W_1 = 62{\cdot}5 \frac{H_2}{H_1}.$$

Alternatively,

$$\frac{W_1}{62{\cdot}5} = \frac{H_2}{H_1}.$$

As the left-hand expression represents the specific gravity of the oil, the right-hand expression does the same. As H_1 was found to be 2·315 ft. and H_2 to be 1·88 ft.,

$$\text{specific gravity of the paraffin} = \frac{1{\cdot}88}{2{\cdot}315} = 0{\cdot}812.$$

The result agrees with the specific gravity as found by the first method.

Density.—When a portion of fluid in motion is under consideration we are concerned with its mass rather than its weight. Different substances have different weights per unit of volume. A cub. ft. of metal, for instance, weighs more than a cub. ft. of wood. Since the mass of a portion of substance is proportional to its weight, different substances have different quantities of mass per unit of volume.

The number of mass units possessed by unit volume of a substance, either liquid or solid, is called the density of that substance.

The number which expresses the density of any given substance, water, for example, depends upon what units of mass and volume are chosen. Taking a cub. cm. as unit volume and regarding a cub. cm. of water as possessing unit mass,

$$\text{density of water} = \frac{1 \text{ mass unit}}{1 \text{ volume unit}} = 1.$$

Similarly, the number 62·5 expresses the density of water when the mass unit is the mass of 1 lb. and 1 cub. ft. is the volume unit.

The most convenient way of dealing with the density of a substance is to compare it with that of water. If a cub. ft. of a substance weighs 125 lb., its weight is twice that of a cub. ft. of water. Consequently the density also is twice that of water; the same fact is otherwise expressed by saying that the density relatively to that of water is 2. The number 2 is also its specific gravity, since 1 cub. ft. of the substance weighs twice as much as 1 cub. ft. of water.

The number which expresses the specific gravity of any substance also expresses its density relatively to that of water, or briefly, its "relative density."

For a given quantity of a particular substance

$$\text{density} = \frac{\text{number of mass units}}{\text{number of volume units}}.$$

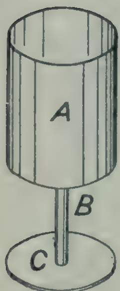

FIG. 28.

Floating Bodies.—A vessel (Fig. 28) constructed of thin metal plate was provided with a stem B; to this was fixed a plate C upon which small slotted weights were placed. The vessel was capable of floating in water when supporting the greatest load applied. Placing the loaded cylinder in the vessel C of Fig. 17, the displaced water was caught and weighed. Removing the cylinder, load was added to the plate below it, after which it was replaced in the vessel C. More water was displaced, which was weighed together with that previously collected. The results obtained after successive repetitions of these operations were as follows:—

Weight of floating body (lb.)	0·38	0·58	0·78	0·98	1·18	1·38	1·58	1·78
Water displaced (lb.) . .	0·39	0·58	0·78	0·99	1·18	1·38	1·58	1·78

The most cursory glance at this table shows that the following statement is justified.

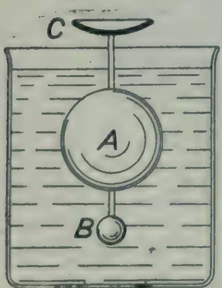

FIG. 29.

When an object floats in water it displaces a quantity of water having a weight equal to its own.

An instrument for determining relative density or specific gravity and termed a *hydrometer* may be made as in Fig. 29. A represents a hollow bulb. A metal ball B is attached to A by a stem. A load pan C is attached to the top of the bulb by a slender stem with a line marked around it. When placed in fresh water, weights placed on the pan C are necessary to cause the bulb to be submerged until the line on the stem is at the water level. Let W_1 be the weight of the instrument together with the load in the pan to produce this effect.

If the instrument is now placed in another kind of liquid either more or less weight at C may be necessary to bring the line on the stem to the level of the liquid, according to whether the liquid is more or less dense than water. In either case let W_2 now be the weight of the instrument together with the load in the pan. Then because W_1 and W_2 are respectively equal to the weights of equal volumes of displaced liquid, and mass is proportional to weight,

$$\text{relative density or specific gravity of the second liquid} = \frac{W_2}{W_1}$$

The weight of a ship is 1,000 tons. When floating in fresh water, a mark, level with the surface of the water, was made on the side of the vessel. Upon proceeding to salt water the mark rose above the level of the water. When 25 tons of cargo were taken on board the mark again became level with the water surface. What is the density of the salt water relatively to that of the fresh water?

Merely by stating the data differently we have:

$$\left.\begin{array}{c}\text{volume of 1,025 tons}\\ \text{of salt water}\end{array}\right\} = \text{volume of 1,000 tons of fresh water.}$$

Hence,

$$\text{density of salt water} = \frac{\text{mass units contained by the salt water}}{\text{mass units contained by the fresh water}}$$

$$\frac{1025}{1000} = 1{\cdot}025.$$

If the ball B at the bottom of the hydrometer (Fig. 29) is replaced by a pan, the specific gravity of a solid substance may be determined. For a substance which floats in water a cage is required to keep the body in the pan. Let W_1 be the weight of the instrument and W_2 the weight of material experimented with. First submerge the instrument in water to the line on the stem by adding a weight w_1 to the upper pan. The substance itself is not yet placed in the lower pan. Since we have a floating body,

$$\text{weight of water displaced by the hydrometer} = W_1 + w_1.$$

Now place the piece of material in the lower pan, and instead of the load w_1 put a weight w_2 in the upper pan so as to bring the line on the stem level with the water surface. Then

$$\left.\begin{array}{l}\text{weight of water displaced by the hydrometer}\\ +\,\text{weight of water displaced by the substance}\end{array}\right\} = W_1 + w_2 + W_2.$$

Subtracting the left and right-hand sides respectively of the former equation from those of the latter,

$$\text{weight of water displaced by the substance} = W_2 + w_2 - w_1.$$

As the value of each term on the right-hand side of the last equation is readily found, it will be easy to solve the following equation:

$$\left.\begin{array}{l}\text{specific gravity of}\\ \quad\text{the substance}\end{array}\right\} = \frac{\text{weight of the substance}}{\text{weight of water displaced by the substance}}.$$

In order to find W_2, the weight of the substance, the instrument may take the place of a balance. Place the substance in the upper pan and add a weight w_3 to bring the line on the stem to the water level. Then

$$W_1 + W_2 + w_3 = W_1 + w_1.$$

Hence,
$$W_2 = w_1 - w_3.$$

A piece of earthenware weighing 14 *grams is placed in the lower pan of a hydrometer. The hydrometer is then submerged in water by adding* 10 *grams to the upper pan. When the earthenware is removed a load of* 17 *grams in the upper pan is required to submerge the hydrometer. What is the specific gravity of the substance?*

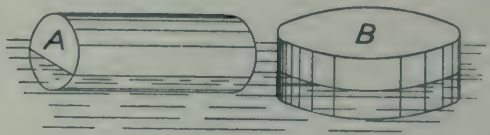

FIG. 30.

$$\left.\begin{array}{l}\text{Weight of water displaced by the}\\ \quad\text{earthenware}\end{array}\right\} \begin{array}{l} = (W_2 + w_2 - w_1)\text{ grams}\\ \quad (14 + 10 - 17)\text{ grams}\\ \qquad 7\text{ grams}\end{array}$$

$$\text{Specific gravity} = \frac{14}{7} = 2.$$

Fig. 30 represents two floating cylinders, one of them (A) with its axis horizontal, the other (B) with its axis vertical. Endeavour to make A float with its axis vertical and B with its axis horizontal. As in both cases failure is inevitable it is obvious that some essential condition must be fulfilled which is only satisfied by A when its axis is horizontal and by B when its axis is vertical.

The centre of gravity of a floating body of any shape is in the same vertical line as the centre of gravity of the liquid displaced.

If this statement were not true then the weight of the floating object would act downward along one vertical line and the force of buoyancy upward along another. The two forces being equal in amount—for a floating object is in equilibrium—a couple would then result. As there is no second couple to balance the first one the conditions of equilibrium would not be satisfied. The truth of the statement must therefore be accepted.

In Fig. 31 let M represent the centre of gravity of the floating cylinder A whilst N is the centre of gravity of the displaced liquid. The point N is termed the *centre of buoyancy*. The points M and N are in the same vertical line. Now apply a couple acting in a vertical plane so as to depress one end of the cylinder. N_1 is now

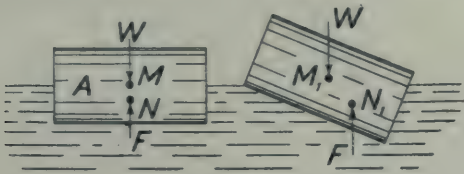

FIG. 31.

the centre of buoyancy, M_1 being the centre of gravity of the cylinder, which, of course, has not moved from its original position within the cylinder. The weight W of the cylinder and the force of buoyancy F form what is termed a "righting" or "restoring" couple which tends to bring the object back to its original position of floating.

Let the same cylinder now be placed in water with its axis vertical, as in Fig. 32, and prevented by some constraint from falling over. Permit the axis to become inclined; M_1 and N_1 having the same meaning as in the preceding instance, the couple produced by the forces F and W now tends to turn the cylinder over on its side. If all constraint is removed the cylinder falls over into a position of stability. Similar conditions apply to the cylinder B in Fig. 30, as in fact they do to objects of any shape whatever. Cylinders were chosen merely for the sake of dealing with some definite shape.

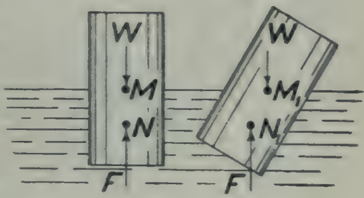

FIG. 32.

In all the preceding experiments and problems in this chapter the liquid is assumed to have no motion. The relationships between the forces exerted upon and by a body of still liquid form the subject termed *Hydrostatics*.

Applications of the Force of Buoyancy.—One of the simplest appliances which depends upon the force of buoyancy for its action is the plumbers' ball-cock illustrated on p. 33 in "Applied Mechanics—First Year." Much more elaborate applications of a float may be found in practice; one such is illustrated in Fig. 33.

A tank is supplied with water by means of a pump driven by the belt A. When the tank is full it is necessary to stop the pump by shifting the belt to the loose pulley. When the tank is nearly empty

the belt must be moved back to the driving pulley. These operations are effected by means of a gear worked by a float D. The arms of the belt fork are separated by a distance which slightly exceeds twice the width of the belt; they are attached to a rod E carrying a "bob-weight" F. The rod is keyed to a rocking spindle to which also is fixed a lever G. The lower end of a rod H is hinged to the lever G; the upper end is hinged to a lever J. The right-hand end of the lever J is hinged to a vertical rod K provided with stops L and M. The object of the link N is to keep the rod K vertical.

The rod carrying the bob-weight F is represented as leaning

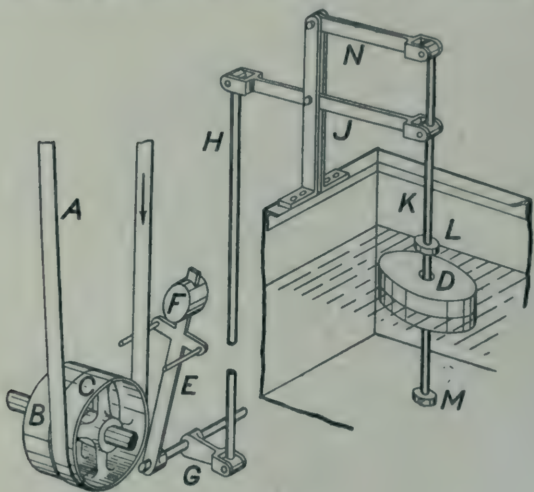

FIG. 33.

over to the right, but when this rod is brought into a vertical position the movable links, rods, and levers, taken as a whole, are balanced. The float D is loaded so that it displaces half as much water when floating as it would do if totally submerged, and is capable of moving with freedom along the rod K between the limits imposed by the stops L and M.

In the given position of the gear the water level is descending and the belt is therefore running on the loose pulley. When the bottom of the float reaches the stop M, the float is arrested and the water level continues to fall until as much of the weight of the load on the float as is necessary to cause the rod E to rise to a vertical position is supported by the stop M. Until this point is reached the right-hand arm of the belt fork will not have touched the belt, but shortly afterward the weight F falls over to the left, causing the belt fork to carry the belt on to the fast pulley. In consequence a sudden descent of the stops L and M takes place. The pump now starts working and the water-level rises until the top of the float

encounters the stop L. The float is then arrested, but the water-level continues to rise until the force of buoyancy is great enough to cause the bob-weight to move over to the right. Shortly after passing the vertical position the rod E falls over to the right and moves the belt back to the position shown in the figure.

Fig. 34 is the section of a steam-trap. Owing to the radiation of heat some of the steam passing along a steam-pipe becomes condensed and it is then most important to eject from the pipe the water so formed. The function of a steam-trap is to do this automatically.

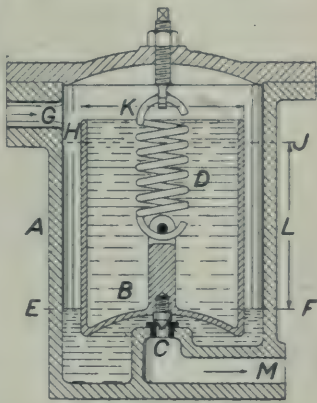

FIG. 34.

A cylindrical cast-iron vessel A is strong enough to resist a steam pressure of the same intensity as that in the steam-pipe. Within the outer vessel is a kind of bucket B supported upon a spring D and having a small valve C below it. The vessel B is always full of water and the water in the outer vessel A is never allowed to fall below a certain level, E F, say. The spring D is adjusted to such a tension that its upward pull is just sufficient to balance the weight it supports—that of the vessel B and the water it contains—when the level of the water outside B is at EF.

Condensed steam from the pipe gravitates into A through the opening G. In consequence, the level of the water, initially at EF, will become higher. When it has risen to some such position as HJ, the force of buoyancy, equal to the weight of a cylinder of water having a diameter K and a length L, is sufficient to balance the pressure of the steam exerted over the area of the hole covered by the valve. Any further rise of the water level results in sufficient buoyancy to overcome the downward force of the steam upon the valve. The bucket and the valve attached will now rise and the pressure of the steam will force the water in the trap through the opening M. As the water level falls the buoyancy decreases until finally the valve once more rests upon its seat.

Referring to Fig. 34, *let K be* 8 *in. and L be* 6 *in. If the water to be ejected from the trap weighs* 60 *lb. per cub. ft. and the diameter of the hole in the valve seat is* 0·3 *in., what is the maximum steam pressure at which the trap may be relied upon to work?*

Force of buoyancy obtained by raising the water-level from EF to HJ } = weight of displaced water.

$$\left(\frac{\pi}{4} 8^2 \times 6 \times \frac{60}{1{,}728}\right) \text{lb.}$$

$$\left(\frac{10}{3} \pi\right) \text{lb.}$$

Let P lb. per sq. in. be the required pressure.

Steam pressure to be overcome = (P × area of valve seat) lb.

$$\left(P \times \frac{\pi}{4}\, 0{\cdot}3^2\right) \text{ lb.}$$

$$(P \times 0{\cdot}0225\pi) \text{ lb.}$$

$$\text{steam pressure} = \text{force of buoyancy}$$

$$(P \times 0{\cdot}0225\,\pi) \text{ lb.} = \left(\frac{10}{3}\,\pi\right) \text{ lb.}$$

$$\therefore \quad P = \frac{10}{3 \times 0{\cdot}0225} \text{ lb. per sq. in.} = 148 \text{ lb. per sq. in.}$$

In manufacturing processes it sometimes happens that two separate liquids of different densities may be required in one tank. If the nature of the liquids does not allow them to mix, there will be a plane of separation between the two. As the respective quantities of the liquids alter, some means will be required to indicate the position of the plane of separation. One such method is illustrated in Fig. 35. The denser liquid is below the level A.

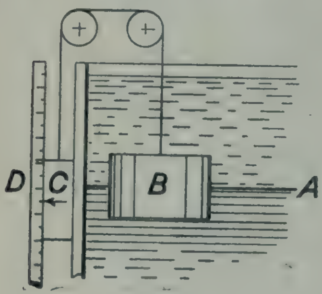

FIG. 35.

A piece of earthenware B is immersed partly in the upper and partly in the lower liquid. A flexible metal cord attached to the earthenware B—which is termed a "float" in spite of the fact that the material may sink in either liquid—is passed over grooved guide pulleys and attached to a counterpoise C. A reading mark on C shows upon a scale D the position of the plane of separation A between the two liquids.

Let the liquid above the level A have a specific gravity of 0·8 *and that below A a specific gravity of* 1·2. *Let the float B have a specific gravity of* 2 *and a volume of* 0·5 *cub. ft. What must the counterpoise at C weigh in order that the float B shall be equally immersed in the two liquids?*

Regard the float as divided by the horizontal plane at A into two equal parts. Consider the equilibrium of each part separately.

$$\left.\begin{array}{l}\text{Weight of liquid dis-}\\ \quad\text{placed by upper}\\ \quad\text{part of float}\end{array}\right\} = 0{\cdot}25 \text{ cub. ft.} \times (0{\cdot}8 \times 62{\cdot}5 \text{ lb.}) = 12{\cdot}5 \text{ lb.}$$

$$\left.\begin{array}{l}\text{Weight of liquid dis-}\\ \quad\text{placed by lower}\\ \quad\text{part of float}\end{array}\right\} = 0{\cdot}25 \text{ cub. ft.} \times (1{\cdot}2 \times 62{\cdot}5 \text{ lb.}) = 18{\cdot}75 \text{ lb.}$$

Force of buoyancy acting upon the whole float	$= 12{\cdot}5$ lb. $+ 18{\cdot}75$ lb. $= 31{\cdot}25$ lb.
Weight of the float	$= 0{\cdot}5$ cub. ft. $\times$ ($2 \times 62{\cdot}5$ lb.) $= 62{\cdot}5$ lb.
Resultant downward force upon the float	$= 62{\cdot}5$ lb. $- 31{\cdot}25$ lb. $= 31{\cdot}25$ lb.

In order to keep the float in equilibrium an upward force of 31·25 lb. must be supplied ; consequently a load weighing 31·25 lb. is required at C.

Let the float be prevented from moving, the plane of separation rising meanwhile until the float is completely immersed in the denser liquid ; the buoyant force now becomes twice 18·75 lb. or

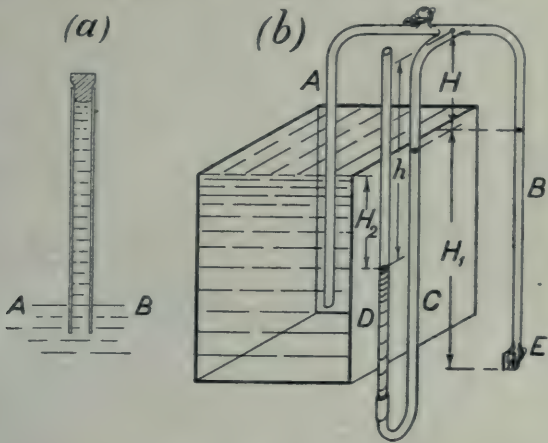

FIG. 36.

37·5 lb. If released the float would move upward because 37·5 lb. represents a greater upward force than the downward force due to the unbalanced part of the weight of the float. A corresponding action is produced if the plane of separation falls instead of rises ; in this case the resultant force will be downward and hence when released the float will fall until equilibrium is established.

The Barometer.—Immerse fully, in a vessel containing water, a glass tube with both ends open. The inside of the tube is at once filled with water. Close one end with a rubber stopper whilst the tube is still submerged. Bring the tube to a vertical position and raise the closed end until nearly all the tube is above the surface of the water. Observe that the tube is still filled with water.

The weight of the water column in the part of the tube above the level AB (Fig. 36) (*a*) must be balanced by an equal upward force due to the pressure of the atmosphere upon the surface of the water

outside the tube. Let the pressure of the atmosphere upon an area equal to that of the section of the tube be P lb.; then the atmosphere exerts a downward pressure of P lb. above the stopper.

If the tube is removed from the tank and the water emptied out, the upward atmospheric pressure upon the lower side of the stopper will also be P lb. Under the conditions shown in Fig. 36 (*a*) if W lb. is the weight of the water column, the upward atmospheric pressure upon the bottom of the stopper is (P — W) lb., because from the upward force of P lb. W lb. must be deducted to balance the weight of the water column.

Consider the equilibrium of a horizontal film of water within the tube at the level AB. An upward pressure of P lb. upon this film, due to the atmosphere, is balanced by the weight of the water W lb., together with a downward reaction of the stopper equal to (P — W) lb. If the tube is made longer, the water column becomes proportionately heavier and eventually a point is reached when W equals P. Using mercury instead of water this point occurs when the column is about 30 in. high.

If the tube is raised yet further the top of the mercury column remains stationary and a vacuum is formed above it. An instrument constructed in this manner is called a *barometer* and is used to determine the intensity of atmospheric pressure at any given time and place.

Suppose that at the same time and place two barometers are read, one of them having a mercury column, the other a water column. If the mercury column is 30 *in. high, what is the height of the water column?*

Refer to Fig. 27, where liquids of different specific gravities are placed in a bent tube. Suppose the column to the left to be of water, the other of mercury. As one column balances the other,

$$\frac{\text{specific gravity of mercury}}{\text{specific gravity of water}} = \frac{H_1}{H_2}.$$

As the specific gravity of mercury is 13·6,

$$H_1 = 13{\cdot}6\ H_2.$$

Since H_2 is stated to be 30 in. or 2·5 ft.,

$$H_1 = (13{\cdot}6 \times 2{\cdot}5)\ \text{ft.} = 34\ \text{ft.}$$

In the case of the two barometers the columns do not actually balance each other, but atmospheric pressure of the same intensity is balanced by each, and hence 34 ft. is the height of the column in the water barometer.

The Syphon.—In Fig. 36 (*b*), AB represents an inverted U-shaped tube to which a branch tube C is connected. The lower end of the branch is turned up and attached to a glass tube D. The upper end of the glass tube and the lower end of the limb A are open. Two

cocks are shown connected to the U-tube; these being open the whole arrangement is submerged in water and the cocks closed before withdrawal. The lower part of the limb A is immersed in a vessel of water, all the rest of the apparatus being outside the vessel.

Open the cock E slowly, until there is a full opening. A continuous stream of water now issues from the end of the tube below the cock, and the water-level in the glass tube falls H_2 ft. below the level of the water in the vessel. Close the cock E slowly and the water levels in the glass tube and the vessel will eventually occupy the same horizontal plane.

In Fig. 36 (*b*) H ft. is the height above the water-level of the horizontal part of the U-tube. The pressure of the water at this part is therefore less than that of the atmosphere by an amount corresponding to the height H ft. when no water is flowing, and to the height h ft. when water issues from the tube. Of the column of water in the limb B, the part H is balanced by the corresponding part in the limb A. The weight of the part H_1 is then supported by the lower part of the closed tube B. Upon opening the cock there is nothing to restrain the column of height H_1 from flowing out, and the atmospheric pressure on the surface of the water in the tank results in an inward flow at the lower end of the limb A.

When the water is flowing, the upper part of the column in B to the extent of h ft. is employed in balancing the resistance to flow offered by the tube together with the weight of the column H in the limb A. Hence the flow of water through the cock E is due to the weight of a column of water of a height $(H + H_1 - h)$ ft. No flow takes place unless this expression has a positive value. Since h becomes greater when the speed of flow is increased it is clear that resistance increases with the speed of flow. The U-tube alone forms what is termed a *syphon*. The rest of the arrangement does not affect the working of the syphon.

EXAMPLES

The weight of 1 *cub. ft. of water is taken as* 62·5 *lb. A gallon of water weighs* 10 *lb.*

31. The pressure of water in a main is to be 25 lb. per sq. in. If the supply is derived from an elevated tank, what should be the height of the water surface in the tank above the level of the main?
32. A circular tank is 40 ft. in diameter. If this contains water to a depth of 18 ft., what pressure, in tons, is exerted on the bottom of the tank?

33. A cylinder for a hydraulic press is cast vertically with the lower end closed. The internal diameter of the cylinder is 15 in. The metal of which it is made weighs 0·26 lb. per cub. in., and when poured into the mould the upper surface of the metal is 3·5 ft. above the lower end of the core which is used to form the hollow space in the cylinder. What upward force is exerted upon the core?

34. A cylindrical tank with its axis vertical has a diameter of 4 ft. The tank is built up of 6 rings of equal width, and its total height is 15 ft. What is the total normal pressure in lb. acting upon the bottom ring when the tank is filled with water?

35. A cylinder of cast-iron, 1 ft. long and 6 in. in diameter, is attached to a balance by a thin cord and lowered into a vessel containing water. When the cylinder is fully submerged the balance indicates 83 lb. Taking the weight of cast-iron as 0·26 lb. per cub. in., how many cub. in. in the casting are taken up by the cavities known as "blow-holes"?

36. One side of a mould for an iron casting is a rectangle, 3 ft. deep by 2 ft. wide. Find the whole pressure on the side of the mould, assuming that the specific gravity of the metal is 7·5.

37. A hollow cylinder with open ends, 10 in. mean diameter, 10 ft. long and $1\frac{1}{4}$ in. thick, is to be cast with its axis vertical. Taking the specific gravity of cast-iron as 7·5, find the pressure on the bottom of the mould when it is full of molten metal.

38. If the cylinder in the preceding question is cast with the lower end closed, what will then be the pressure on the bottom of the mould?

39. The rectangular side of a tank is constructed of two plates, the joint being horizontal. The upper plate is 5 ft. deep, and when the tank is full the pressure on this plate is equal to that on the lower one. How deep is the tank?

40. A cylindrical tank, 2 ft. in diameter, contains but is not filled with water. A cylinder 1 ft. long and 1 ft. in diameter is supported on a cord and lowered into the water until it is submerged. No water leaves the tank during this operation. What increase of pressure upon the base of the tank is caused by submerging the cylinder?

41. A vessel weighs 2,000 tons, but requires an additional 30 tons at high water to bring it down to the same draught as at low water. Find the ratio of the specific gravities of the water at high and low water respectively.

42. A cylinder 10 in. long is made of a material with a specific gravity of 0·6. Another cylinder of identical shape and size is formed of a material with a specific gravity of 8. The two cylinders are joined end to end and immersed in water. The compound cylinder is maintained in a horizontal position by an upward pull along a cord. How far from the outer end of the denser part of the cylinder must the cord be situated?

43. A lock-gate is 10 ft. high and 14 ft. wide. On one side the water is level with the upper edge of the gate; on the other side the water-level is 6 ft. below the upper edge. What is the resultant pressure upon the gate and how far below the upper edge is the point through which the resultant pressure acts?

44. A flood-gate is 2 ft. wide and 3 ft. deep. A vertical post is fixed to the centre of the gate and extends 6 ft. above the upper edge. The gate is capable of turning about an axis coincident with its upper edge. Water on one side of the gate is level with the upper edge. What horizontal pull applied at the top of the post will cause the gate to turn against the pressure of the water?

45. An oil-drum is 2 ft. long and 2 ft. in diameter. The drum being filled with water, its cylindrical axis is inclined to the ground at 45°. A small vent-hole is made at the highest point in the drum. What pressure is exerted by the water upon the lower circular end of the drum?

46. A circular plunger, weighing 12 lb. and 4 in. in diameter, is free to slide in a vertical cylinder. The space between the under surface of the plunger and the bottom of the cylinder is in communication with a column of oil, 2·75 ft. high. The pressure due to this column is sufficient to counterpoise the weight of the cylinder. What is the specific gravity of the oil?

47. A cylindrical metal rod 2 in. in diameter is supported vertically by a spring-balance. A portion of the rod being lowered into water, the balance reads 2 lb. less than it did before. How many inches is the lower end of the rod below the surface of the water?

48. The plan of a tank is a square of which the length of the side is 4 ft. The depth is 6 ft. A liquid having a specific gravity of 1·2 is contained in the lower part of the tank, its upper surface being 2 ft. above the base of the tank. Another liquid with a specific gravity of 0·78, fills the rest of the space in the tank. What pressure is exerted upon one side of the tank?

49. A barometer tube contains a column of glycerine which is 330 in. high when a column of mercury in another barometer tube is 30·6 in. high. Taking the specific gravity of mercury as 13·6 what is the specific gravity of the glycerine?

50. A steam trap constructed as shown in Fig. 34 has a valve of $\frac{3}{8}$ in. diameter, the diameter of the counterpoised vessel being 6 in. The steam pressure being 90 lb. per sq. in., what rise of the water-level in ft. above the level of equilibrium at no pressure must occur in order that the valve may open?

(*Take the weight of the water as* 62 *lb. per cub. ft.*)

CHAPTER III

Pumps.—A pump is a machine for transferring fluid from one place to another. Most commonly the fluid is water which has either to be raised from a low to a higher level or forced under pressure into some vessel—a boiler, for instance. The essential parts of a single-acting pump are shown in Fig. 37.

A cylindrical plunger A is moved up and down by means of a rod

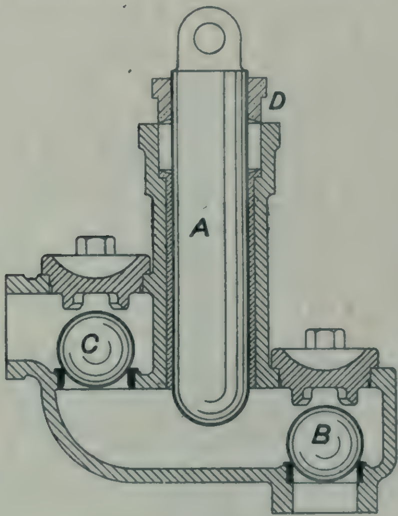

FIG. 37.

worked by a crank or an eccentric. Beneath the plunger is a chamber provided with two circular holes. One of these is closed by a valve B termed the suction valve, the other by a valve C termed the delivery valve. Beneath B, a length of pipe termed the suction pipe is so fixed to the pump that air cannot leak in at the joints. The lower end of the pipe is situated below the surface of a body of water in a tank or well. A pipe termed the delivery pipe is fixed to the pump to carry away the liquid passing through the hole below C.

Suppose the pump chamber and the suction pipe to be full of water whilst the plunger is rising. If the level of the bottom of the plunger is less than 34 ft. above the water-level in the suction tank, a vacuum is not formed; as the plunger rises the pressure of the atmosphere forces water up the suction pipe; the valve B then lifts

and water enters the chamber. When the plunger descends, the valve B closes and the water leaving the chamber is compelled to raise the valve C and pass along the delivery pipe. In order to prevent leakage past the plunger a stuffing-box and gland D are necessary.

The height of the suction valve above the level of the water in the suction well must not exceed 34 ft., or the water cannot enter the pump chamber. For various reasons this height, or "suction ift," never approaches 34 ft., and under some circumstances it must be zero even.

The diameter of the plunger of a single-acting pump is 2 *in. The plunger is operated by a crank* 3 *in. long which turns round* 50 *times per min. The pump is placed* 15 *ft. above the level of the water in a well, and forces water to a height of* 120 *ft. above the level of the water in the well. What is the effective H.P. of the pump and what force is exerted upon the plunger when water issues from the pump?*

$$\text{Length of stroke} = 6 \text{ in.} = 0{\cdot}5 \text{ ft.}$$

$$\text{Sectional area of plunger} = \left(\frac{\pi}{4} \times \frac{4}{144}\right) \text{sq. ft.} = 0{\cdot}0218 \text{ sq. ft.}$$

$$\text{Volume of water raised per min.} = (0{\cdot}5 \times 0{\cdot}0218 \times 50) \text{ cub. ft.}$$
$$= 0{\cdot}545 \text{ cub. ft.}$$

$$\text{Lb. of water raised per min.} = 62{\cdot}5 \times 0{\cdot}545 = 34.$$

$$\text{Energy per min.} = (120 \text{ ft.} \times 34 \text{ lb.}) = 4{,}080 \text{ ft.-lb.}$$

$$\text{Effective H.P.} = \frac{4{,}080}{33{,}000} = 0{\cdot}1235.$$

The position of the pump with respect to the suction and delivery levels does not influence the amount of energy required per min., but the higher the pump is placed above the suction level the more work will be performed during the suction stroke and the less during the delivery stroke. As the delivery level is (120 — 15) ft. or 105 ft. above the plunger, the force exerted on the plunger must be that due to a column of water 105 ft. high acting upon a surface of 0·0218 sq. ft. Hence,

$$\text{force on the plunger} = (105 \times 0{\cdot}0218 \times 62{\cdot}5) \text{ lb.} = 143 \text{ lb.}$$

Machines worked by Water Pressure.—Machines for shearing metal plates, riveting the shells of boilers, forging iron and steel, compressing bulky materials such as hay and cotton, etc., are commonly operated by water supplied under great pressure. All such contrivances are termed *hydraulic machines.* (The word *hydraulics* denotes the mechanics of liquids in motion.) Very large forces are exerted through short distances at a low speed in the case of most hydraulic machines. Commonly the water is supplied from a main pipe fed by pumps which produce the necessary pressure in the water. The pressure varies from about 800 lb., to several tons, per sq. in.

A hydraulic lifting-jack (Fig. 38) is often employed when heavy loads are to be raised by human effort. At the top of a bored cylinder A an enclosed cistern B is formed. Within the cistern is a small pump C, shown in section by the side of the main part of the figure. A spindle passes through a stuffing-box in the side of the cistern. To the outside end of the spindle a lever handle D is attached. Within the cistern a short lever arm E is fixed to the spindle. A cylindrical "ram" F is provided at the top with a leather packing to prevent leakage. When the jack is in use, water is taken in at the suction valve G from the cistern and forced

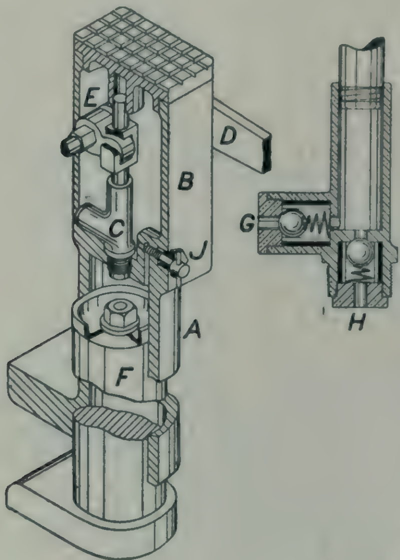

FIG. 38.

through the delivery valve H into the space between the top of the ram and the bottom of the cistern. In consequence, the cylinder A, all the parts connected to it and any load supported by the cistern cover or the foot projecting from the cylinder, must rise as the volume of water below the cistern becomes greater. When the valve J is opened the water returns to the cistern and the load descends.

The ram of a hydraulic jack has a diameter of 3 in.; that of the pump plunger is $\frac{3}{4}$ in. The short arm of the lever is $1\frac{1}{2}$ in. long and the lever handle is 20 in. long. What is the velocity ratio of this machine?

Assume that when the plunger makes a downward stroke of 1 in. the load rises M in.

Volume of water leaving the pump } = volume of water entering the cylinder A.

Length of stroke × section of plunger } = rise of load × section of ram.

$$\left(1 \times \frac{\pi}{4}(0{\cdot}75)^2\right) \text{ sq. in.} = \left(M \times \frac{\pi}{4} 3^2\right) \text{ sq. in.}$$

$$M = \frac{(0{\cdot}75)^2}{3^2} = 0{\cdot}0624 \text{ in.}$$

Let N in. be the movement of the end of the long arm, the point at which the effort is applied. Then

$$\frac{\text{length of long arm}}{\text{length of short arm}} = \frac{\text{movement of effort point}}{\text{movement of plunger}},$$

or,

$$\frac{20 \text{ in.}}{1{\cdot}5 \text{ in.}} = \frac{N \text{ in.}}{1 \text{ in.}}$$

and

$$N = 13{\cdot}\dot{3} \text{ in.}$$

$$\text{velocity ratio} = \frac{N}{M} = \frac{13{\cdot}\dot{3}}{0{\cdot}0624} = 214 \text{ (approx.)}.$$

The Hydraulic Accumulator.—Hydraulic machines are generally arranged in groups of which each machine may be started or stopped independently. Water for the whole group may be supplied by one pump. If the water were pumped directly into the main it would be necessary to start the pump every time a machine was put into action. If a second machine were started subsequently the pump would have to work more rapidly in order to supply the extra water. Moreover, the pump would have to be controlled from different and often distant points. A pump works most efficiently as a rule at one particular speed, and hence it would be not only a difficult and complicated but a wasteful method to put into practice if the machines were supplied directly by the pump. For these and other reasons, the pump discharges into an accumulator, an appliance illustrated in Fig. 39.

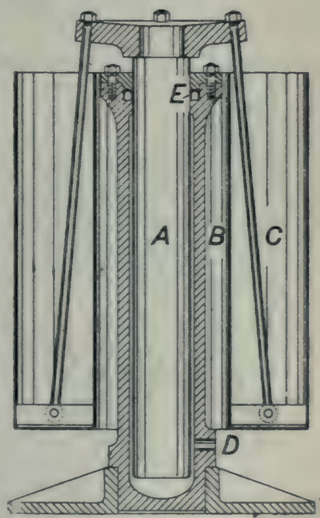

FIG. 39.

An accurately turned cylindrical ram A moves up and down a hollow cylinder B. An annular vessel C, loaded with heavy material, is supported by the ram and moves up and down with it. The inside of the cylinder is connected to the water main by a pipe

fixed at D. When more water is delivered by the pump than is flowing along the water main, water enters the cylinder and the ram ascends; when more water is taken from the main than is delivered by the pump water flows from the cylinder and the ram descends. At the upper limit of its movement a tappet operated by the moving parts causes the pump to stop working. After the ram has descended a certain distance, the pump is caused to start again by the operation of another tappet. Leakage of water is prevented by the packing at E.

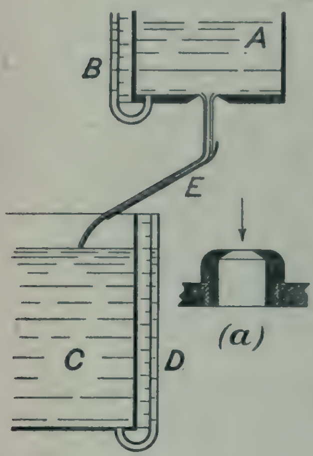

FIG. 40.

The intensity of pressure of the water depends upon the sectional area of and the load upon the ram. The amount of energy that may be stored is proportional to the load upon and the length of stroke of the ram. The main function of this appliance is not—in spite of its name—to store energy, but to enable sudden and irregular demands for energy to be met whilst the pump which supplies it is working at a uniform rate.

The ram of an accumulator is 6 *in. in diameter. The intensity of pressure of the water in the hydraulic main is* 1,000 *lb. per sq. in. The stroke of the ram is* 6 *ft. What load (including the weight of the ram itself) is required upon the ram and what amount of energy may be stored in the accumulator?*

$$\text{Sectional area of ram} = \frac{\pi}{4} 6^2 \text{ sq. in.} = 28{\cdot}27 \text{ sq. in.}$$

$$\text{Load required} = (1{,}000 \times 28{\cdot}27) \text{ lb.} = 28{,}270 \text{ lb.}$$
$$\text{Energy stored} = 6 \text{ ft.} \times 28{,}270 \text{ lb.} = 169{,}620 \text{ ft.-lb.}$$

Flow of Water through an Orifice.—If a tank A (Fig. 40), preferably much deeper than the diagram suggests, has in its base a sharp-edged circular hole or *orifice*, then any water contained by the tank issues through the hole in a jet which becomes very nearly cylindrical in shape at a short distance below the bottom of the tank. The greatest diameter of the finely tapering part of the jet is plainly less than that of the hole. The rate of supply of water to the tank may be arranged to equal that at which it flows through the hole. A glass tube B enables the height H ft. of the water-level, which is termed the *head*, to be observed upon a length scale.

Our object is to determine the relationship between the height H and the rate of flow through the orifice. In order to measure the

outflow it is convenient to make use of another tank C. A glass tube D adjoins a scale which indicates the amount of water contained by the tank C when the water surface is at any particular level. The scale is formed by pouring measured quantities of water into the tank and marking against the resulting water-level in the tube the total amount of water added.

Knowing the initial quantity of water, if any, in the tank C, divert the jet by means of the deflector E into that tank, observing the instant at which this is done. Keeping the surface of the water in A at the same level, note the head H. When a considerable quantity of water has collected, allow the jet to flow once more

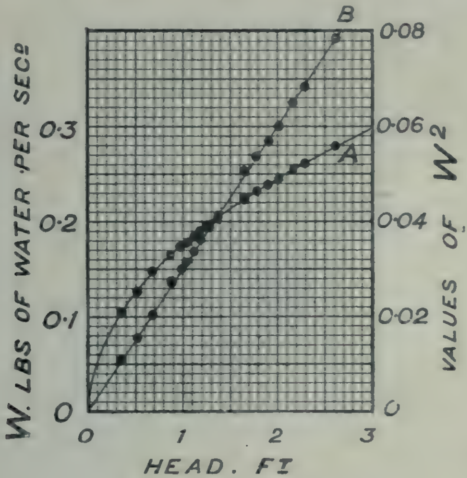

FIG. 41.

into the drain, observing the exact instant when this commences. We are now able to find out how much water, which may be denoted W lb., flowed through the hole in one second for a particular head of H ft. Repeating these operations with different heads and using an orifice of 0·02625 ft. diameter formed in a brass fitting as shown at (*a*) Fig. 40, the following table resulted :—

Head (ft.) .	0·345	0·51	0·69	0·895	0·982	1·05	1·11	1·195	1·295
W (lb. per sec.)	0·104	0·125	0·147	0·166	0·173	0·177	0·183	0·19	0·197
W^2 . .	0·0109	0·156	0·0216	0·0275	0·03	0·0313	0·0336	0·0361	0·0388

Head (ft.) .	1·392	1·665	1·795	1·925	2·025	2·185	2·3	2·625
W (lb. per sec.)	0·203	0·225	0·232	0·2387	0·245	0·255	0·261	0·28
W^2 . .	0·0412	0·0506	0·0538	0·057	0·06	0·065	0·9682	0·0784

Plotting the first two columns of numbers gives the graph A (Fig. 41), which resembles either of those in Fig. 6 ; and for the same reasons as are given on p. 16 the squares of the numbers along the vertical axis are now plotted. This gives the straight-line graph B (Fig. 41), of which the equation is

$$W^2 = 0{\cdot}03\ H.$$

Taking the square root of each side of the equation,

$$W = 0{\cdot}173\ \sqrt{H}.$$

By means of two pointed screws as in Fig. 42, the diameter of the jet is measured as nearly as possible at the point where it first approaches the cylindrical shape. For the experiment of which the results are given, this diameter proved to be 0·021 ft., from which the area of the section of the jet at the same place is $0{\cdot}7854\ (0{\cdot}021)^2$ sq. ft. or 0·000347 sq. ft. The volume of water flowing past the section in question equals the product of the speed of flow and the sectional area of the jet.

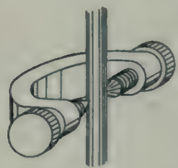

FIG. 42.

Taking 62·4 lb. as the weight of a cub. ft. of water,

$$\frac{W}{62{\cdot}4} = \text{number of cub. ft. per sec.}$$

Hence,

$$\left.\begin{array}{c}\text{speed of flow}\\ \text{(ft. per sec.)}\end{array}\right\} = \frac{\text{cub. ft. per sec.}}{\text{sec. area of jet (sq. ft.)}} = \frac{W}{62{\cdot}4 \times 0{\cdot}000347} = 46{\cdot}25\ W.$$

The equation to the graph B (Fig. 41) enables us to express the last equation as follows :—

$$\text{speed of flow (ft. per sec.)} = 46{\cdot}25 \times 0{\cdot}173\sqrt{H} = 8\sqrt{H}.$$

Choosing from the table a head of, say, 1·295 ft, and denoting the speed of flow as V ft. per sec., we have

$$V = 8\sqrt{1{\cdot}295} = 9{\cdot}1 \text{ ft. per sec.}$$

The weight of water flowing through the hole at the stated head is 0·197 lb. per sec. (see table on p. 67), and therefore the K.E. of the jet is

$$\frac{WV^2}{2\,g} = \frac{0{\cdot}197 \times (9{\cdot}1)^2}{64{\cdot}4} = 0{\cdot}253 \text{ ft.-lb. per sec.}$$

The speed of descent through the tank is very small indeed, so that practically no K.E. is acquired until the water reaches the neighbourhood of the orifice. As each portion of water loses

energy of position during its descent it acquires an equal quantity of energy due to the pressure of the water surrounding it, but the pressure diminishes when the speed increases, and having passed the orifice the only pressure existing is that of the atmosphere; hence the energy of position lost by a descent of 1·295 ft. is converted into K.E. The slight difference of level between the plane of the orifice and the section where the speed of flow is measured may be ignored at present without causing any serious error; if H ft. represents the descent of W lb. of water and V ft. per sec. be the speed thereby acquired, we may expect our measurements to be in agreement with the equation :—

$$\left.\begin{array}{l}\text{Initial energy of position of}\\ \quad\text{W lb. of water}\end{array}\right\} = \text{final K.E. of W lb. of water,}$$

or,

$$WH = \frac{WV^2}{2g}.$$

Referring to the table on p. 67 we have W equal to 0·197 lb. when H is 1·295 ft. Hence,

$$WH = (0{\cdot}197 \times 1{\cdot}295) \text{ ft.-lb.} = 0{\cdot}2525 \text{ ft.-lb.}$$

As the corresponding K.E. has just been found to be 0·253 ft.-lb., the smallness of the discrepancy between the two results tends to support the view that the energy of position lost during the descent of the water reappears as K.E. in the jet.

The discrepancy between the two quantities of energy increases, however, when a correction is made, for the diameter of the jet was measured at a point 0·075 ft. below the plane of the orifice; the actual descent of the water to that point is therefore (1·295 + 0·075) ft. or 1·37 ft., which gives the initial energy of the jet as equal to 1·37 ft. × 0·197 lb. or 0·27 ft.-lb. As our measurements account only for 0·253 ft.-lb. of K.E., either the measurements are inaccurate or energy has been lost during the descent of the water.

Coefficient of Velocity.—From the preceding equation, viz.,

$$WH = \frac{WV^2}{2g}$$

we have

$$V^2 = 2gH$$

or

$$V = \sqrt{2gH}.$$

The speed of the jet as calculated from the last equation may be termed the "ideal speed of flow." When the surface of the water is 1·295 ft. above the base of the tank, the speed of flow is found by observation (see p. 68) to be 9·1 ft. per sec.; but the actual head of water above the section where this speed occurs is 1·37 ft. At this section therefore,

$$\text{ideal speed of flow (ft. per sec.)} = \sqrt{2gH} = \sqrt{64{\cdot}4 \times 1{\cdot}37} = 9{\cdot}4.$$

Does the discrepancy between 9·1 and 9·4 represent experimental error? Very careful investigation has shown that a drop of speed, attributable to internal friction in the jet, actually does occur. Compare the two speeds thus :—

$$\frac{\text{actual speed of flow}}{\text{ideal speed of flow}} = \frac{9{\cdot}1}{9{\cdot}4} = 0{\cdot}97.$$

This ratio is called the *Coefficient of Velocity*, and the value obtained agrees with that which is commonly accepted; the value changes slightly, however, under different conditions. The discrepancy between the initial energy of position, 0·27 ft.-lb., with respect to the section where the speed is measured, and the K.E. of the jet at the same place, viz. 0·253 ft.-lb., mainly represents therefore an actual loss of energy due to internal friction of the jet.

Coefficient of Discharge.—Suppose that the water issued from the orifice in a cylindrical jet of the same diameter as the orifice. Suppose also that the ideal speed of flow is attained. If then D ft. is the common diameter of the orifice and jet, W_1 lb. of water per sec. is the flow resulting from the assumptions, and taking 62·4 lb. as the weight of a cub. ft. of water,

$$W_1 = 62{\cdot}4\,V\,\frac{\pi}{4}\,D^2 = 62{\cdot}4\sqrt{2gH} \times \frac{\pi}{4}\,D^2 = 49\sqrt{2gH} \times D^2.$$

For an orifice of the diameter used, that is, 0·02625 ft., and with H equal to 1·295 ft.,

$$W_1\ (\text{lb. per sec.}) = 62{\cdot}4 \times 8{\cdot}04 \times \sqrt{1{\cdot}295} \times 0{\cdot}7854 \times (0{\cdot}02625)^2$$
$$= 0{\cdot}308.$$

The actual discharge from an orifice depends not only upon its area and the head of water above it but upon the shapes of its horizontal

FIG. 43.

and vertical sections. Each of these may be formed in various ways, some of the latter being given in Fig. 43.

W_1, the ideal discharge, as found above, is taken as a standard of comparison for the actual discharge at the same head from any orifice having an area equal to that of a circle of diameter D. The actual discharge W from a sharp-edged circular orifice of 0·02625 ft. diameter has been found to be 0·197 lb. per sec. when H is 1·295 ft. Comparing the two results,

$$\frac{W}{W_1} = \frac{0{\cdot}197}{0{\cdot}308} = 0{\cdot}64.$$

The resulting ratio is termed the *Coefficient of Discharge of the orifice*; the value given is for a particular head, viz., 1·295 ft., but is found to alter only slightly for considerable variations of the head. This statement does not apply to very low heads, for the conditions of flow are not then the same as for high ones. The coefficient of discharge also alters to a slight extent when the diameter of the orifice is changed.

The water flowing through an orifice of which the coefficient of discharge and the area are, respectively, 0·6 *and* 0·8 *sq. in., is* 50 *gallons per min. What is the head (H) in ft.?*

Actual discharge = ideal discharge × coefficient of discharge.

$$\left(\frac{50 \times 10}{60}\right) \text{lb. per. sec.} = 62{\cdot}4\sqrt{2gH} \times \frac{0{\cdot}8}{144} \times 0{\cdot}6$$

$$\sqrt{2gH} = \frac{500 \times 144}{60 \times 62{\cdot}4 \times 0{\cdot}8 \times 0{\cdot}6} = \frac{7{,}200}{179{\cdot}5}$$

$$2gH = \left(\frac{7{,}200}{179{\cdot}5}\right)^2 = 1{,}600$$

$$H = \frac{1{,}600}{64{\cdot}4} = 24{\cdot}9 \text{ ft.}$$

Coefficient of Contraction.—The ratio of the area of the jet section (at the level where the jet begins to taper finely) to the area of the orifice is called the *Coefficient of Contraction.* Assume that a jet issues from an orifice with no diminution of section, but at a speed V_1 equal to the actual speed of the real jet. Since 0·97, the coefficient of velocity, is the ratio of the actual speed to the ideal speed,

$$V_1 = 0{\cdot}97\sqrt{2gH}.$$

Modifying the final expression for W_1 the ideal rate of discharge given on p. 70,

ideal discharge at speed V_1 (lb. per sec.) $= 0{\cdot}97 \times 49\sqrt{2gH} \times D^2$.

If the quantity resulting from the expressión for the ideal discharge as given on p. 70 is multiplied by the coefficient of discharge, we obtain the actual discharge. Hence, the coefficient of discharge having been obtained as 0·64,

$$\frac{\text{actual discharge}}{\text{ideal discharge at equal speed}} = \frac{0{\cdot}64 \times 49\sqrt{2gH} \times D^2}{0{\cdot}97 \times 49\sqrt{2gH} \times D^2} = 0{\cdot}66.$$

If two jets have equal speeds, their respective rates of discharge are proportional to their sectional areas; consequently the number just found, 0·66, is the coefficient of contraction. Since the numbers

obtained for the coefficients of velocity and discharge are not rigidly applicable under all circumstances, neither will be the number found for the coefficient of contraction.

Momentum of a Jet of Water.—In Fig. 44 two quadrant-shaped plates about 4 in. long are shown attached to a small block A. Two

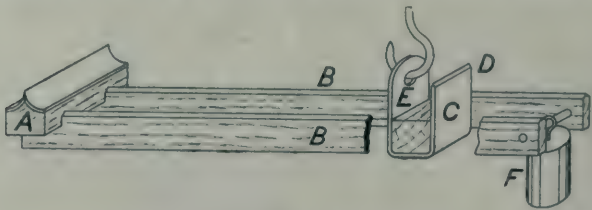

FIG. 44.

parallel strips BB connect the block A to a bent strip of metal C. At D a knife-edge is formed; at E is a knife-edged hole in which the hook of a spring-balance is inserted. The knife-edge to the right being supported upon a metal plate, a downward force acting at the centre of the block A has ten times the leverage of the supporting force measured by the spring-balance. Hence a vertical force acting in a line through the sharp edge common to the two curved plates is readily found. The load F partially counterpoises the weight of the parts on the left of the knife-edge.

Arrange the apparatus as in Fig. 45 so that a vertical jet of water

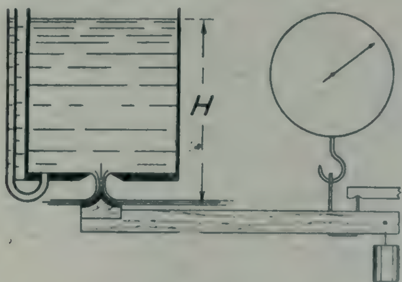

FIG. 45.

is equally divided by the sharp edge. The two parts are deflected and shoot off horizontally from the plate. Whilst this is happening a pull is indicated by the balance. At the point of meeting the curved plates, the jet possesses vertical momentum, resulting mainly from the pressure exerted by the surrounding water as it acquired speed in leaving the tank; but partially due to the force of gravity acting upon the water as it moves down from the orifice to the plates. As the water leaves the plates, half of it possesses momentum in a horizontal direction toward the right; the other

half possesses horizontal momentum toward the left. In consequence the horizontal momentum of the water, as a whole, is nothing. The vertical momentum of the water is now also nothing. The resultant pressure exerted upon the plates is due to the loss of momentum by the jet. The K.E. and therefore the speed of the jet also are due to a total head H (see Fig. 45), but the discharge corresponds to a head slightly less owing to the distance between the respective levels of the orifice and the curved plates. Ignoring this difference of level, with the same orifice as before the vertical pressure on the plate as found by the balance was 0·13 lb. when H was 3 ft. Using the equations given on pp. 68 and 71,

$$W = 0{\cdot}173\sqrt{H} = 0{\cdot}173\sqrt{3} = 0{\cdot}3 \text{ lb. per sec.}$$

$$V = 0{\cdot}97\sqrt{2gH} = 0{\cdot}97\sqrt{64{\cdot}4 \times 3} = 13{\cdot}3 \text{ ft. per sec.}$$

$$\text{Momentum per sec.} = \frac{WV}{g} = \frac{0{\cdot}3 \times 13{\cdot}3}{32{\cdot}2} = 0{\cdot}124 \text{ unit.}$$

Rate of change of momentum (units of momentum per sec.) is force ; hence 0·124 lb. is the calculated pressure upon the plates, a result which agrees with the observed force of 0·13 lb. Further observations and calculations are as follows :—

Head (ft.)	2·75	2·57	2·42	2·3	2·05	1·65
Pressure (lb.)	0·12	0·11	0·105	0·1	0·09	0·07
W (lb. per sec.)	0·287	0·277	0·269	0·262	0·248	0·222
V (ft. per sec.)	12·9	12·47	12·1	11·8	11·14	10·0
$\frac{WV}{g}$	0·115	0·107	0·101	0·096	0·086	0·069

Comparing the numbers in the second line with those in the fifth, a fairly close agreement is observable.

At a head of 10 ft., water flows through a hole in the bottom of a tank. The area of the hole is A sq. ft. and the coefficient of discharge is 0·6. The jet impinges on a quadrant shaped plate and shoots off horizontally. The downward pressure caused on the plate is 2 lb. How much is A ?

$$\left.\begin{matrix}\text{Discharge (lb.}\\ \text{per sec.)}\end{matrix}\right\} = \sqrt{2gH} \times 0{\cdot}6\,A \times 62{\cdot}4 = W.$$

$$\left.\begin{matrix}\text{Momentum}\\ \text{per sec.}\end{matrix}\right\} = \frac{WV}{g} = \frac{\sqrt{2gH} \times 37{\cdot}44\,A \times 0{\cdot}97\sqrt{2gH}}{g} = 72{\cdot}7\,HA.$$

Force (lb.) = change of momentum per sec.

$$2 = 72{\cdot}7 \times 10 \times A.$$

$$A = \left(\frac{2}{727}\right) \text{sq. ft.} = 0{\cdot}00275 \text{ sq. ft.}$$

When the curved plates, instead of being quadrants, are semicircular as in Fig. 46, the divided jet shoots off in a vertical direction. If the loss by friction is negligible, then each part of a jet reaching the plates with a downward speed of V ft. per sec. leaves with an upward speed of the same amount. Regarding the downward momentum as positive, the upward momentum is negative.

$$\underset{\text{(per sec.)}}{\text{Change of momentum}} = \underset{\text{(per sec.)}}{\text{initial momentum}} - \underset{\text{(per sec.)}}{\text{final momentum}}$$

$$\frac{WV}{g} \qquad - \left(\frac{-WV}{g}\right).$$

That is, the total change of momentum per sec. is $\frac{2WV}{g}$ and the pressure upon the plate is now twice as much as when the water left the plate horizontally.

If curved plates are attached to the rim of a wheel (see Fig. 149) change of momentum is manifested as an impulse which is capable of urging the wheel round against the resistance of a torque. Energy will therefore be transferred from the water to the wheel. A contrivance for utilising the K.E. of a jet in this manner is termed an *impulse wheel.* Water-wheels and turbines of the type in which the water does not alter in pressure whilst acting upon the wheel are examples of impulse wheels. A further consideration of these contrivances is deferred until the subject of "normal acceleration" has been treated.

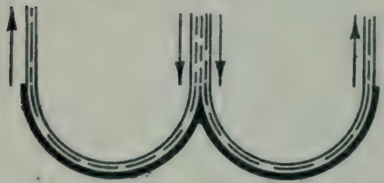

FIG. 46.

EXAMPLES.

51. A hydraulic press is actuated by a hand-pump. The diameter of the ram of the press is 6 in., that of the plunger of the pump is $1\frac{1}{2}$ in.; the length of the short arm of the pump lever is 2 in., and that of the long one is 18 in. Find the velocity ratio of this machine.
52. A hydraulic jack has a ram 4 in. in diameter. The plunger of the pump is 1 in. in diameter. The short arm of the pump-lever is 2 in. and the long arm of the same is 18 in. long. If the end of the long arm is depressed through a distance of 6 in. how far will the load rise?
53. A hydraulic jack has a ram of 3 in. diameter. The pump has a plunger $\frac{3}{4}$ in. in diameter. The short arm of the lever is

2 in. long, the long arm is 16 in. long. When lifting a load of W lb. the pressure at the end of the handle is 24 lb., and the M.E. of the jack is 80 per cent. How much is W ?

54. A hydraulic accumulator has a stroke of 10 ft. and a ram of 20 in. diameter. The total load on the ram being 130 tons, what is the water pressure in lb. per sq. in., and what is the greatest amount of energy that may be stored?

55. The ram of an accumulator is 10 in. in diameter and the water pressure is 1,000 lb. per sq. in.

(*a*) What is the load on the ram (lb.) ?

(*b*) What is the stroke of the ram (ft.) if it is capable of actuating for 20 minutes a machine which absorbs $\frac{1}{2}$ H.P. ?

56. The ram of an accumulator has a diameter of 20 in. and is loaded so as to cause a water pressure of 1,000 lb. per sq. in. At what rate in H.P. does energy issue from the accumulator when the ram falls at a speed of 3 ft. per min. ?

57. The ram of a hydraulic accumulator is 6 in. in diameter. The water pressure is 800 lb. per sq. in. How much energy is stored in this apparatus when the load has been lifted 6 ft. ?

58. An accumulator ram is 12 in. in diameter, and is loaded with 50 tons. What amount of energy in ft.-tons is obtainable from a cub. ft. of water in this system, assuming an efficiency of 80 per cent. ?

59. A hydraulic crane is supplied with water at a pressure of 700 lb. per sq. in., of which 2 cub. ft. are required in order to raise a load of 4 tons through 12 ft. What is the mechanical efficiency of the crane ?

60. A circular hole in the bottom of a tank discharges 50 lb. of water per min. when the surface of the water in the tank is 3·6 ft. above the bottom of the tank. What discharge will occur when the level of the water is 5 ft. above the bottom of the tank ?

61. A hole in the bottom of a tank permits water to flow through it at the rate of 25 lb. per min. when the level of the water surface is 3·85 ft. above the hole. What must the height be to cause the flow to be three times as much ?

62. A hole in a tank, 1 sq. in. in area, is situated 5 ft. below the water-level. Water flows through the hole at the rate of 0·08725 cub. ft. per sec. Find the coefficient of discharge of the hole.

63. The coefficient of discharge of an orifice is 0·76. If the area of this orifice is 2·5 sq. in., how many cub. ft. of water per min. will be discharged through it at a head of 12 ft. ?

64. A vessel 4 ft. high is kept full of water whilst a jet flows from a hole in its base. If the coefficient of discharge of the hole is 0·5 and 4 cub. ft. of water issue in 30 min., what is the area of the hole in sq. in. ?

CHAPTER IV

Stress.—If the upper end of a vertical rod is fixed to a support and a load is attached to the lower end, the weight of the load is exerted along the whole length of the rod. At any horizontal section of the rod, two balanced vertical forces, each equal to the weight of the load, are distributed over the section, and each is directed away from the section. Such a pair of forces, tending to pull the rod apart, produces a *tensile stress.* If a load is supported on a column, a force equal to the weight of the load acts downwards upon any transverse section and an equal force acts upwards, towards it. The column is then subjected to *compressive or crushing stress.*

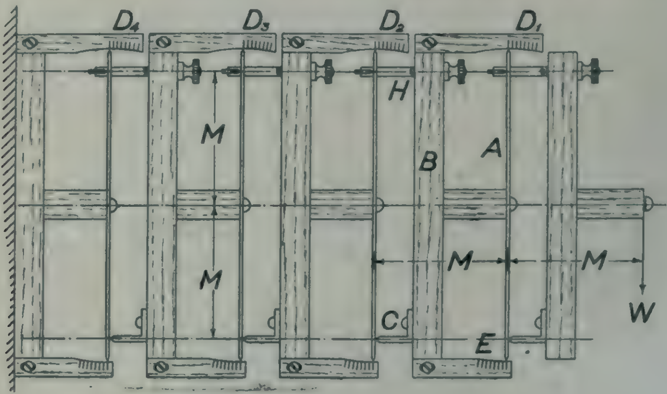

Fig. 47.

It is convenient to use the word "stress" to denote the amount of force distributed over unit area of section of a piece of material subjected to tension, compression, or shearing; it may then be regarded as an abbreviation of the phrase "intensity of stress," since the word "stress" is used to denote the mutual action between adjoining molecules of a piece of material when a pair of balanced forces tends to produce distortion or rupture.

Forces causing Bending in a Cantilever.—A cantilever consists of a horizontal piece of metal or timber, with one end rigidly fixed to a support, the other end being free; its function is to support a load or loads. The effect of the loading is to set up stresses within the cantilever and cause it to bend. For the purpose of investigating the action of the bending forces an apparatus may be built up as shown in Fig. 47.

Five elements are connected in such a way as to form what is essentially a cantilever. Excepting the elements at either end, each is constructed as follows: To each end of a T-shaped piece of wood B a strip is fixed upon which a scale—D_1 at the top and E at the bottom—is formed to indicate horizontal forces acting upon the element when a load W is supported at the end of the cantilever. A screwed rod H, with a milled nut at one end and a knife-edged plate brazed into a slit at the other, fits closely in a hole bored in the T-piece. A knife-edged bracket C is screwed to the lower end of the T-piece. A steel spring A—about 1 in. wide and $\frac{1}{16}$ in. thick—is fixed rigidly to the horizontal part of the T.

Before connecting the elements each is fastened to a support; equal and opposite forces—each force being a multiple of 1 lb.—are then applied to the spring along the lines in which the knife-edges are situated. In consequence, the upper part of the spring is bent to the right, the lower part to the left. A mark is made opposite to the end of the spring and the amount of each force is registered. A sufficient number of regularly progressing forces being applied, the required scale results. When the elements are connected, their weights cause "sagging" to take place. By means of the milled nuts the sag may be taken up. The zero points of the scales will not now be level with the ends of the springs, but they may be adjusted by means of slots in the scale strips. In the apparatus made by the writer the dimension M was 3·5 in. and an increase of 1 lb. in the force acting on either knife-edge was indicated on the scale by about 1·5 mm. The T-piece to the extreme left was screwed to a support.

A load W of 2 lb. was now applied as shown. The resulting indications of the scales were as follows:—

Scale	D_1	D_2	D_3	D_4
Force (lb.)	1	2	3	4

Each reading applies to either the top or the bottom scale. The results are shown in a diagram (Fig. 48).

In order to prevent the element at the extreme right of Fig. 47 from moving downward under the action of the load W, an upward reaction of 2 lb. is exerted upon the supporting screw by the spring A (Fig. 47) attached to the next element. Hence a clockwise couple having a moment of 2 lb. × 3·5 in. or 7 lb.-in. is acting on the first element. Since the scales on the adjoining element indicate 1 lb. each, an anti-clockwise couple with a moment of 1 lb. × 7 in. or 7 lb.-in. is also exerted on the first element by virtue of two horizontal forces, one a pull upon the screw at the top and the other a thrust upon the knife-edged bracket at the bottom. The four forces acting on the element are in equilibrium. The weight

of the element is balanced by an upward reaction of the supporting spring A.

If the element supporting the load, together with the one adjoining, is now regarded as a single object, the stresses exerted between its two parts may be left out of consideration in dealing with the equilibrium of the forces acting on the whole object. The two vertical forces exerted upon the supposed single object are still 2 lb. each, but the couple which they form has in this case a moment of 2 lb. × 7 in. or 14 lb.-in. Upon reference to Fig. 48 it will be seen that the balancing couple due to the horizontal forces indicated by the scales is of exactly the same amount.

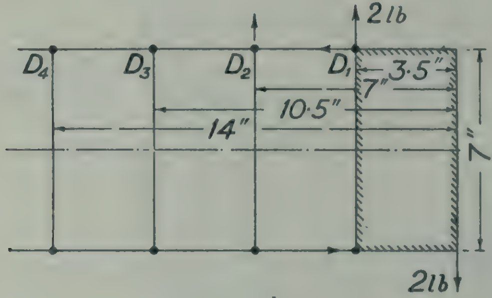

FIG. 48.

Three, and then four, elements may be taken together and treated in the same manner. In each case a pair of couples of equal moment are found to be keeping in equilibrium the portion of the cantilever under consideration. Measuring the horizontal couples at each vertical section coincident with the initial position of one of the springs (A, Fig. 47), and disregarding the lateral displacement due to the bending of the springs, we obtain the first column of the following table :—

Moment of couple (lb.-in.) . . .	7	14	21	28
Force (lb.)	1	2	3	4
Length (in.)	3·5	7	10·5	14

The second column gives the horizontal force which, acting on the left-hand end of the portion of the cantilever considered, produces tension in the upper part and compression in the lower part ; the length of the part considered is given in the third column. Plotting the numbers in the second and third columns, the graph A (Fig. 49) results. The plotted points lie in a straight line. Arguing from the results yielded by the apparatus and the graph A, we may

conclude that in an actual cantilever there is a resultant tensile stress along some line, as P (Fig. 50), and a resultant compressive stress along a line Q. The amount of each stress is directly proportional to the distance of the vertical section where it is measured,

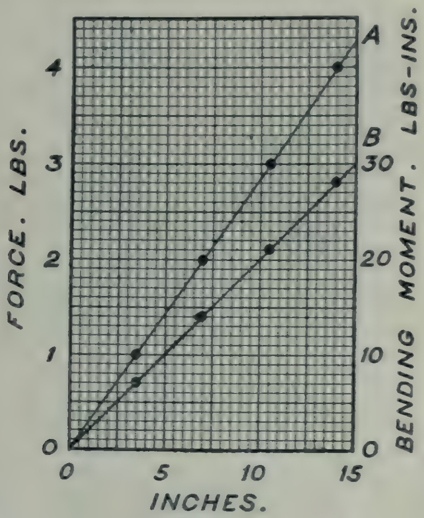

FIG. 49.

from that end of the cantilever where the load is applied. The graph B (Fig. 49) is obtained by plotting the numbers in the first and third columns of the last table.

The moment of the couple formed by the vertical forces is called a bending moment; the moment of the other couple is termed the moment of resistance.

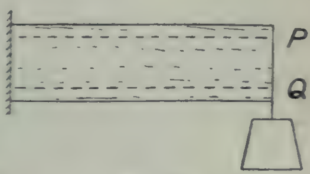

FIG. 50.

There is no essential difference between an actual cantilever and the apparatus of Fig. 47; the same force action occurs in each, but in the apparatus provision is made for measuring it at certain points.

Skin Stress in a Loaded Beam.—Fig. 51 represents a beam of uniform rectangular section supported at the points A and B and subjected to equal loads W at the sections C and D. The lengths N are equal, but are unrelated to the length M. Since the beam is symmetrically loaded the upward reactions at A and B are each equal to W. The part of the beam lying between A and C or D and B is therefore subjected to the action of a couple having a moment W × N. Each of the two parts requires a balancing couple, which can only be due to forces acting along the intermediate part of the

beam, and distributed over the whole area of a vertical section. For the present these forces may be regarded as concentrated along the lines of action of their resultants.

Choose a vertical section, as at E, anywhere along the length M,

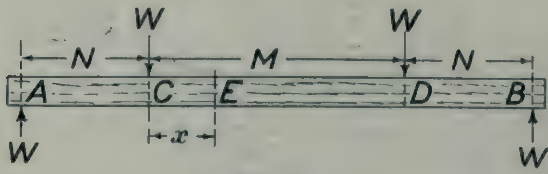

FIG. 51.

and estimate the bending moment at that section ; in terms of the forces acting on the beam to the left of the section, we have

$$\text{bending moment at E} = W(N + x) - Wx = WN.$$

The same result is obtained if the forces to the right of the section are considered instead, and the equation shows that the bending moment is the same for all sections in the length M.

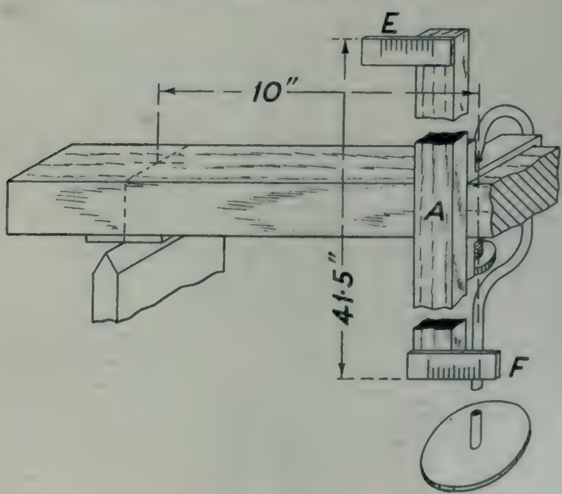

FIG. 52.

Before applying any load, attach to the beam at C and D two long vertical strips of wood as A in Fig. 52. The dimension 10 in. in this figure corresponds to N in Fig. 51. By means of load pillars, apply equal weights at the sections C and D (Fig. 51). The ends of the strips now move together at the top and apart at the bottom. The beam becomes curved with its concave surface

at the top, the shape of the part M changing from a rectangular prism to a portion of an annulus. This effect is due to the shortening of the upper fibres by compression and the lengthening of the lower fibres by tension. Hence there is compressive stress in the upper part and tensile stress in the lower part of the middle portion of the beam. Of the forces producing these stresses, those exerted *by* the part M form two couples, each having a moment equal to WN and balancing the bending couple acting on either end-piece of the beam.

A small portion of the middle part of the beam is shown in Fig. 53. The top layer shortens from AB to A_1B_1, whilst the bottom layer lengthens from CD to C_1D_1. If the rate of stretching is equal to the rate of compression, the length of the central horizontal layer of the beam remains unaltered and the change of length of any other horizontal layer is proportional to its distance from the central layer. Whether at the centre or not there must be an unstressed layer at the level where compression ends and tension

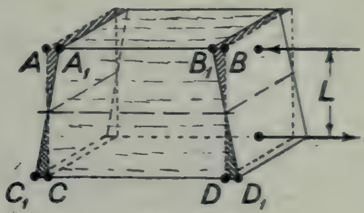

FIG. 53.

begins. This layer has no finite thickness and is termed the *neutral plane or surface. The intersection of the neutral plane with any vertical cross-section of the beam forms a neutral axis.*

"Strain" is the ratio of a change of length to the initial length; hence the strain which occurs at the top or the bottom layer is greater than at any other place between that layer and the neutral plane. The strain in any layer is proportional to the stress to which that layer is subjected. The stress in the top or bottom layer is termed *skin stress*, and may be determined experimentally.

To the ends of the vertical strips attach short scales of length (E and F, Fig. 52). Equal loads being applied at C and D (Fig. 51), the upper scales move inward and the lower ones outward. Two stretched vertical threads fixed in front of the scales enable these movements to be measured. As the scales are required to ascertain how far the upper ends of the strips move toward each other and the lower ends move apart, any pair of indications of the two upper scales, as also of the two lower ones, are added together. An experiment performed upon a beam of dry black walnut yielded the following results :—

Movement together of upper scales (in.)	0·07	0·14	0·21	0·27	0·33	0·40	0·46	0·52
Movement apart of lower scales (in.)	0·07	0·13	0·20	0·25	0·32	0·38	0·44	0·51
Load on each pillar (lb.)	2	4	6	8	10	12	14	16

Plotting the scale movements against the load on either pillar—compression being treated as negative extension—we obtain the graph (Fig. 54). The straight line is characteristic of an elastic

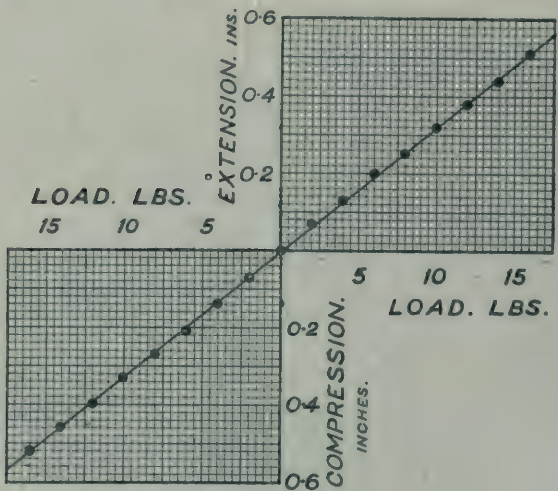

FIG. 54.

material and shows that the rate of compression is equal to the rate of extension. The slope of the graph is 0·0322, which indicates that, for every lb. of load on each of the load pillars, a movement—of the upper scales together and of the lower ones apart—of 0·0322 in. took place. The graph shows also that the movement together of the two ends of the central layer of the beam—due to the curvature produced by bending—is too small to be taken into account ; for otherwise the graph would deviate sensibly from a straight line.

The depth of the beam was 0·83 in. and the total length of each of the strips carrying the scales was 41·5 in. The distance from the centre of the beam to either the upper or the lower scale is therefore fifty times half the depth of the beam. Consequently the movement together of the upper scales is fifty times the amount of compression that occurs in the top layer of fibres of the beam. Similarly,

the movement apart of the lower scales is fifty times the extension of the bottom layer of fibres. Hence,

compression of top layer or extension of bottom layer of fibres per lb. of load on each pillar $\left.\right\} = \dfrac{0{\cdot}0322}{50}$ in. $= 0{\cdot}000644$ in.

The distance M in Fig. 51 being 20 in.,

strain produced by 1 lb. of load on each pillar (compression or tension) $\left.\right\} = \dfrac{0{\cdot}000644 \text{ in.}}{20 \text{ in.}} = 0{\cdot}0000322$

In order to determine the intensity of stress corresponding to a given strain it is necessary to know the value of Young's Modulus for the material. This was ascertained by putting the beam into a testing machine and subjecting it to successive measured tensile forces. Fig. 55 shows the arrangement adopted. A and B represent the parts of the machine to which the piece of timber H—which is no longer a beam but a test-piece—is attached. The inclined surfaces which are in contact with metal packing pieces—one of which is marked C—were formed after the conclusion of the preceding experiment. A micrometer caliper D was attached to a strip of wood E by means of a slot in the lower part of the strip; a screw F was used to fix the two rigidly together. The upper end of the strip E was attached to the wooden test-piece. The movable part of the micrometer caliper was brought up, after each application of load, to the rounded head of a screw inserted in the strip of wood G. The effective length of the test-piece was 30 in. The results obtained were as follows:—

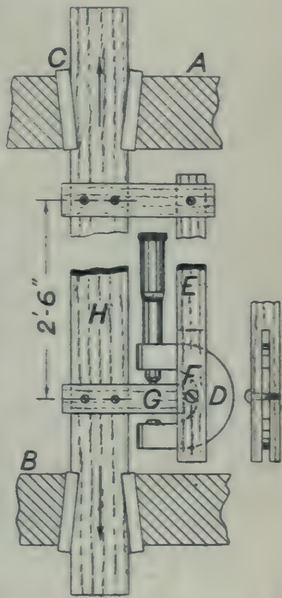

FIG. 55.

Load (lb.) . . .	100	200	300	400	500	600
Micrometer reading (in.) .	0·9332	0·932	0·931	0·9295	0·928	0·927

Load (lb.) . . .	700	800	900	1000	1100	1200
Micrometer reading (in.) .	0·9255	0·924	0·9226	0·9215	0·92	0·919

Plotting these numbers, the graph (Fig. 56) is obtained; its slope shows that a force of 75,300 lb. would be required to extend the test-piece 1 in., provided the same rate of extension were main-

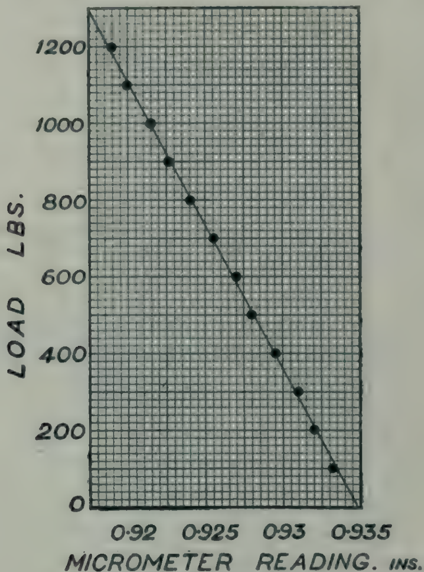

FIG. 56.

tained. The sectional area of the test-piece was $(1{\cdot}18 \times 0{\cdot}83)$ sq. in. or 0·98 sq. in. Hence,

$$E = \frac{\text{stress}}{\text{strain}} = \frac{\text{load}}{\text{extension}} \times \frac{\text{length}}{\text{sectional area}} = 75{,}300 \times \frac{30}{0{\cdot}98}$$
$$= 2{,}300{,}000 \text{ lb. per sq. in.}$$

The bending experiment shows that a strain of 0·0000322 is produced in the top and bottom layers of fibres by an addition of 1 lb. to each of the load pillars. Hence,

$$\left.\begin{array}{l}\text{tensile stress in bottom}\\ \text{layer or compressive}\\ \text{stress in top layer per}\\ \text{lb. of load on each}\\ \text{pillar}\end{array}\right\} = E \times \text{strain} = 2{,}300{,}000 \times 0{\cdot}0000322$$
$$= 74 \text{ lb. per sq. in.}$$

The length N was 10 in. and therefore the addition of each lb. of load on the pillar resulted in 10 lb.-in. of bending moment. Hence

$$\text{skin stress per lb.-in. of bending moment} = \frac{74}{10} = 7{\cdot}4 \text{ lb. per sq. in.}$$

When statical principles are applied, no other data than the shape and dimensions of the beam section is necessary. In any given portion of the middle part of the beam (Fig. 51) the length of a horizontal layer is increased or reduced by stress action to an extent which is proportional to the distance of the layer from the neutral plane (see Fig. 53). Hence the strain and consequently the stress also are likewise proportional to that distance.

Referring to p. 45 we find that the intensity of fluid pressure upon the vertical side of a tank varies in exactly the same way, and that the resultant pressure acts through a point called the centre of pressure situated at $\frac{2}{3}$ of the distance between the levels of no pressure and maximum pressure away from the former level. The point in any vertical section of the beam through which the resultant compressive force acts is therefore equivalent to the centre of pressure and is situated at a distance from the neutral axis equal to $\frac{2}{3}$ of half the depth of the beam. The line of action of the resultant tensile force is similarly obtained. Hence if D denotes the depth of the beam, the distance between the line of action of the resultant of the compressive forces and that of the stretching forces is $\frac{2}{3}$ D.

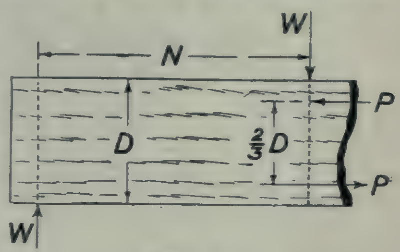

FIG. 57.

Fig. 57 indicates the forces maintaining the equilibrium of one of the end parts of the beam, P being the resultant force of tension or compression exerted by the intermediate portion of the beam. Equating the couples acting upon the end part,

$$WN = \frac{2}{3}\,PD$$

$$\therefore \quad P = \frac{3WN}{2D}.$$

Let B represent the breadth of the beam. Each force P being spread over half the vertical section of the beam (see Fig. 53),

$$\text{average stress (tension or compression)} = \frac{P}{\frac{1}{2}DB} = \frac{2P}{DB}.$$

Since the stress varies uniformly over the half section, from nothing at the neutral surface to the maximum or skin stress, the latter is twice the average stress. Hence,

$$\text{maximum or skin stress} = \frac{4P}{DB}.$$

Substituting the value found for P,

$$\text{skin stress} = \frac{4}{DB} \times \frac{3WN}{2\,D} = \frac{6WN}{BD^2}.$$

Applying this result to the beam used for the preceding experiment in which B = 1·18 in., D = 0·83 in., N = 10 in., W = 1 lb.,

$$\text{skin stress} = \frac{6 \times 1 \times 10}{1{\cdot}18 \times (0{\cdot}83)^2} = 73{\cdot}8 \text{ lb. per sq. in.}$$

This result being due to a bending couple of 10 lb.-in., for a couple of 1 lb.-in. the skin stress is 7·38 lb. per sq. in. As the experimental determination of the skin stress gave 7·4 lb. per sq. in., the agreement between the two results may be taken as satisfactory evidence of the correctness of the reasoning employed.

The equation expressing the skin stress may be given in a more general form. For the bending moment WN let the more usual symbol M be substituted; let f denote the intensity of the skin stress. Then

$$f = \frac{6M}{BD^2}$$

or

$$M = \frac{fBD^2}{6}$$

Let

$$\frac{BD^2}{6} = Z$$

Then

$$M = fZ.$$

Z represents a number which is called the *modulus of the section*; its value in the present example applies only to an *elastic* beam or cantilever of rectangular cross-section which, when supporting vertical loads, has two sides of that section horizontal, the other sides of the section being, of course, vertical.

A balk of timber 11 *in. deep is divided by horizontal cuts into three pieces, of which the depths are respectively* 2 *in.*, 4 *in. and* 5 *in. What is the strength of the initial balk, regarded as a beam, compared to the combined strengths of the three separate parts?*

For a single depth of 11 in. three separate depths are substituted. Nothing else changes, and since the strength varies as the square of the depth,

$$\frac{\text{strength of initial beam}}{\text{strength of the three parts}} = \frac{(11)^2}{2^2 + 4^2 + 5^2} = \frac{121}{45} = 2{\cdot}69.$$

The strength of a beam or a cantilever is utilised as far as possible when it is bent sufficiently to develop the greatest safe skin stress. As this occurs only in the layers furthest removed from the neutral plane the resisting capacity of any other layer is not fully developed, the less so the nearer it is situated to the neutral plane. It is advantageous therefore to arrange the material of a metal beam as shown in Fig. 58, which represents the section of a rolled steel joist. The parts A and B are termed the flanges; the part C is the web.

The bulk of the metal, being in the flanges, is situated as far as possible from the neutral plane. The average stress in the flanges therefore differs little from the skin stress. The resultant forces along the upper and the lower parts of the joist respectively being as widely separated as they can be, the moment of resistance is large in relation to the amount of material used.

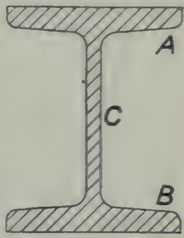

FIG. 58.

If we neglect the inconsiderable effect of the web in resisting the bending couple and regard the average stress in each flange as distributed uniformly over the section, the approximate moment of resistance is readily calculated; the exact moment of resistance is slightly larger.

The flanges of a plate girder are each 8 in. wide and $\frac{3}{4}$ in. thick. The web is 12 in. deep and $\frac{1}{2}$ in. thick. What is the approximate moment of resistance of the section?

Let f lb. per sq. in. denote the safe stress for each flange.

$$\begin{aligned}\text{Strength of either flange} &= \text{area of flange section} \times \text{stress}\\ &= 8 \text{ in.} \times \tfrac{3}{4} \times f\\ &= 6f \text{ lb.}\end{aligned}$$

$$\left.\begin{array}{l}\text{Distance between centres of}\\ \quad\text{flanges}\end{array}\right\} = 12 \text{ in.} + \tfrac{3}{4} \text{ in.}$$

$$\text{Moment of resistance} = (6f) \text{ lb.} \times 12\tfrac{3}{4} \text{ in.} = 76\tfrac{1}{2} f \text{ lb. in.}$$

Bending Moment Diagrams.—The bending moment acting at any section of a beam or a cantilever loaded in any given manner is

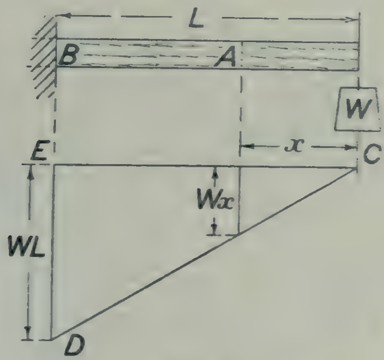

FIG. 59.

shown graphically by a *bending moment diagram*. In a cantilever supporting a single load W, as in Fig. 59, the maximum bending moment occurs at the section B and amounts to WL units of bending moment. From a base line EC set off an intercept of a vertical

line through B representing WL to a stated scale of bending moment. There is no bending moment at the place where the load is applied, and consequently the corresponding intercept at C has no length.

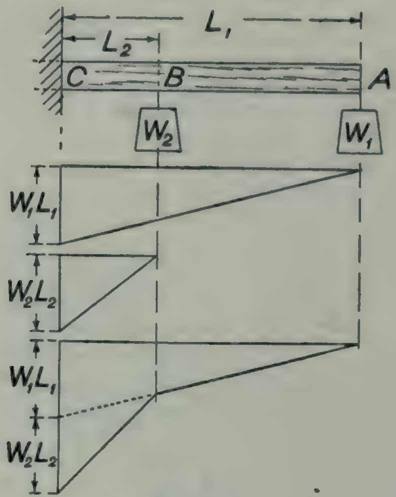

FIG. 60.

Refer to the graph B (Fig. 49) and it is manifest that the bending moment varies uniformly between the two values obtained. Hence if a straight line joins the points C and D, a vertical line drawn through the cantilever at any section A shows, by the length of the

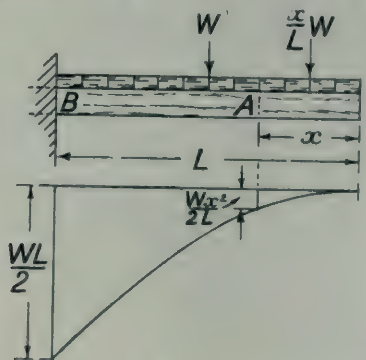

FIG. 61.

intercept between EC and CD, the bending moment at that section, viz: Wx units of bending moment.

Bending moments are regarded as positive when the resulting strain produces a concave surface in the upper part of the material,

Since the upper part of the cantilever in Fig. 59 will become convex the bending moments at all sections are negative and the intercepts by which they are represented are accordingly set off below the base line.

If the cantilever supports two loads as in Fig. 60, two diagrams may be drawn, one for each load. These may be combined to represent the resultant effect.

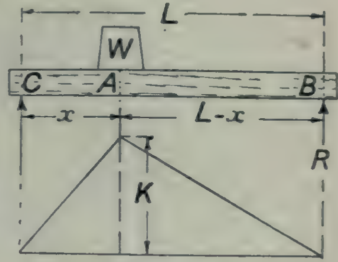

FIG. 62.

Suppose a load W to be uniformly distributed over the length of the cantilever (Fig. 61), the material composing the load consisting not of a single rigid piece but of numerous parts. The resultant weight of all the parts acts along a vertical line at a distance equal to $\frac{1}{2}$L from either end of the cantilever. Hence the maximum bending moment, which occurs at the section B, is equal to $\frac{1}{2}$WL. The weight supported by the part of the cantilever to the right of any such section as A is $\frac{x}{L}$ W. The resultant line of action of this weight is at a distance $\frac{1}{2}x$ to the right of the section A. Hence the bending moment at A is $\frac{x}{L}W \times \frac{x}{2}$ or $\frac{Wx^2}{2L}$. Replacing the constant factors of the last expression by the symbol k, the bending moment is kx^2. Hence the bending moment diagram is a parabola. If w represents the load per unit of length, wx is equivalent to $\frac{x}{L}$ W. The bending moment at A may therefore be expressed as $\frac{wx^2}{2}$.

For a beam supporting a single concentrated load, the maximum bending moment occurs at the section where the load is placed. If R (Fig. 62) is the reaction of the right-hand support, B, by equating the turning moments about C,

$$RL = Wx$$

$$\therefore \quad R = \frac{x}{L}W$$

bending moment at section A $= \frac{x}{L}W \times (L - x) = Wx - \frac{Wx^2}{L}$.

This is shown by the intercept K. If the load is placed at the centre, x becomes $\frac{L}{2}$ and the bending moment is then $\frac{WL}{4}$.

If there are two loads, W_1 and W_2 (as in Fig. 63), the diagram A applies to W_1 and the diagram B to W_2. The diagram C showing the resultant effect is formed by adding the intercepts K_1 and K_2 to obtain M, and K_3 and K_4 to obtain Q. If W_1 is equal to W_2 and

the length N is equal to the length N_1, as in the case illustrated in Fig. 51, the intercepts M_1 and Q_1 are equal and all the intervening intercepts are of the same height.

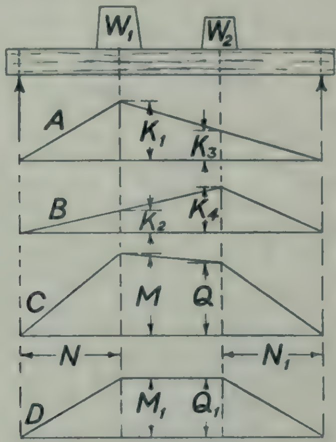

FIG. 63.

A metal rod of rectangular section rests upon two supports placed 8 ft. apart. The breadth and depth of the section respectively are 2 in. and 1·5 in. What load, situated at 3 ft. from one of the supports, produces a skin stress of 12,000 lb. per sq. in.?

W lb. = required load.
R lb. = reaction of the further support.
M lb.-in. = bending moment.

Equating turning moments with respect to the support near the load,

$$\text{R lb.} \times 8 \text{ ft.} = \text{W lb.} \times 3 \text{ ft.}$$

$$\text{R} = \frac{3}{8}\text{W}.$$

$$\text{M} = 5\ \text{R lb.-ft.} = \frac{15}{8}\text{W lb.-ft.} = \frac{180}{8}\text{W lb.-in.}$$

$$\text{M lb.-in.} = \frac{\text{skin stress} \times \text{breadth} \times (\text{depth})^2}{6}$$

$$= \frac{12{,}000 \times 2 \times (1{\cdot}5)^2}{6} = 9{,}000 \text{ lb.-in.}$$

$$\frac{180}{8}\text{W} = 9{,}000$$

$$\text{W} = \frac{9{,}000 \times 8}{180} \text{ lb.} = 400 \text{ lb.}$$

Shearing.—Fig. 64 represents a block of indiarubber with a metal plate cemented to each face. One of these is fixed in a vertical plane and the other supports a load pillar. On the face of the front plate a vernier scale is engraved. When a load is applied, the front plate moves downward past a fixed scale of length, and the indiarubber is distorted into a shape which is indicated by the sloping dotted lines. The results obtained by applying successive loads are here given :—

Load (lb.) . . .	0	5	10	15	20	25	30	35
Deflection (in.) . .	0	0·02	0·045	0·07	0·095	0·12	0·145	0·17

Plotting these numbers we obtain a graph (Fig. 65) of which the slope is 200. The initial intercept of the vertical axis is attributable to an error of 0·005 in. in the zero reading of the scale, and may be disregarded. If the movement of that surface of the block attached to the front plate continued to be proportional to the applied load until a deflection of 1 in. was attained the load would then be 200 lb. The dimensions of the block were A 12·5 in., B 3·4 in. and C 1 in. The area of contact between the plate and the block is A × C and therefore amounts to 12·5 sq. in.

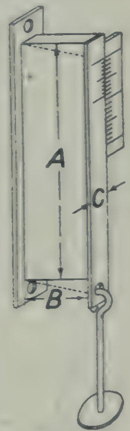

FIG. 64.

Consider the two parts into which the block is divided by any plane of section parallel to either plate. The front part tends to slide down over the rear part, and this tendency produces *shearing* or *tangential stress.* The load on the pillar is distributed over 12·5 sq. in., and hence,

$$\text{shearing stress (lb. per sq. in.)} = \frac{\text{load on the pillar (lb.)}}{12{\cdot}5 \text{ sq. in.}}.$$

The ratio of the relative displacement of two parallel planes in a piece of material subjected to shearing to the distance between these planes is termed shear strain.

Referring to the table of results it is seen that when a load of 25 lb. is applied, the relative displacement of the plates is 0·12 in. The distance between the plates being 3·4 in.,

$$\text{shear strain} = \frac{0{\cdot}12 \text{ in.}}{3{\cdot}4 \text{ in.}} = 0{\cdot}0353,$$

and

$$\text{shearing stress} = \frac{25 \text{ lb.}}{12{\cdot}5 \text{ sq. in.}} = 2 \text{ lb. per sq. in.}$$

The ratio $\dfrac{\text{shearing stress}}{\text{shear strain}}$ *is called the modulus, or coefficient, of rigidity, and is usually denoted "G." The same ratio is sometimes termed the coefficient of transverse elasticity.*

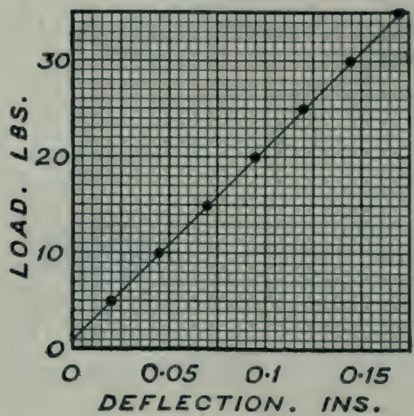

FIG. 65.

The slope of the graph (Fig. 65) expresses the ratio $\dfrac{\text{load}}{\text{deflection}}$.

Hence, if B represents the distance between the planes of which the deflection is the relative displacement,

$$G = \frac{\text{shearing stress}}{\text{shear strain}} = \frac{\dfrac{\text{load}}{\text{area}}}{\dfrac{\text{deflection}}{\text{B}}} = \frac{\text{load}}{\text{deflection}} \times \frac{\text{B}}{\text{area}}$$

$$= \frac{200 \text{ lb.}}{1 \text{ in.}} \times \frac{3{\cdot}4 \text{ in.}}{12{\cdot}5 \text{ sq. in.}} = 54{\cdot}4 \text{ lb. per sq. in.}$$

Shear in Cantilevers and Beams.—Refer to Fig. 47, which shows an apparatus for illustrating the relationships of the forces acting upon definite portions of a cantilever. Each element of the apparatus is prevented from moving downward by the resistance to shear of the supporting screw. In an actual cantilever, loaded as shown in Fig. 59, a shearing force equal to W is distributed over any vertical transverse section of the cantilever. If loaded as in Fig. 60 the shearing force is W_1 for all sections between A and B, and $W_1 + W_2$ for any section between B and C.

The shearing force in any beam or cantilever is at any section equal to the algebraical sum of all the vertical forces acting to the right or left hand of that section.

In Fig. 51, for example, the shearing force acting at any vertical section in either of the lengths N is W lb., whilst at any section in the length M it is nothing.

Clamp together—not too tightly—at a number of points, several thin strips as in Fig. 66. Supporting at the ends the object thus

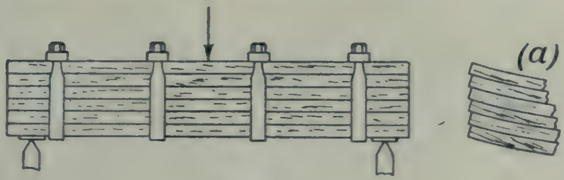

FIG. 66.

built up, apply sufficient load to cause considerable curvature. If initially all the left-hand ends, say, were in a single vertical plane, it is found after bending takes place that the end of each strip occupies a separate plane as shown at (*a*). If the clamps are tightened, a greater load is necessary in order to produce the same deflection as before. It is obvious that sliding has taken place along the initially horizontal planes of contact between the slices. This tendency to slide is existent when a solid beam or cantilever is bent, and results in shearing stress along horizontal planes.

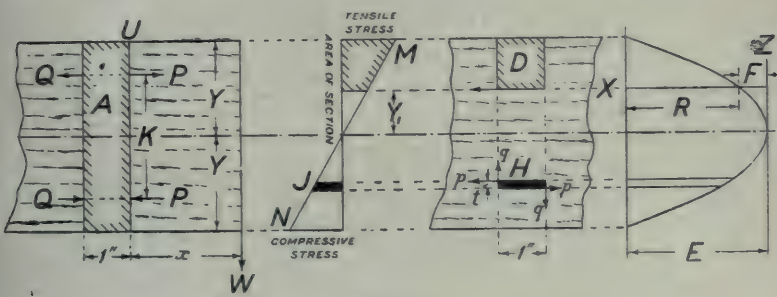

FIG. 67.

Fig. 67 represents part of a cantilever of rectangular cross section, a single load of W lb. being supported at the end. The portion of length x is kept in equilibrium by two equal and opposite couples Wx and PK. The upward force in the first couple is due to shearing resistance along the vertical section U; the second couple is formed by the horizontal forces of tension and compression which have already been discussed (p. 79). (The arrow-heads on the lines of the forces P are not concerned with the couple PK, as they denote the directions of the forces exerted upon the shaded part A of the cantilever.) Similarly two equal and opposite couples,

W (x + 1 in.) and QK maintain equilibrium in the portion of length x + 1 in. From the two equations,

$$Wx = PK$$
$$W(x + 1 \text{ in.}) = QK,$$

we have by subtraction

$$W = K(Q - P).$$

Hence,

$$Q - P = \frac{W}{K}$$

This result being independent of the length x we have therefore upon any vertical slice which is 1 in. thick—such as the one marked A—a resultant horizontal force equal to Q — P or $\frac{W}{K}$ acting toward the left on the upper half; the same amount acts toward the right on the lower half. Each resultant force is the summation of forces distributed over the portion of the vertical section above or below the neutral axis. A horizontal line extending from the vertical to the sloping side of either of the triangles M or N represents the intensity of the stress at the level where it is drawn. The area of either triangle represents the force Q — P which produces shearing along a portion of the neutral surface having a length of 1 in.; if the cantilever is B in. wide, this portion of surface has an area of B sq. in., and

$$\left.\begin{array}{l}\text{shearing stress (lb. per sq. in.)}\\ \text{at the neutral plane}\end{array}\right\} = \frac{Q - P}{B} = \frac{W}{KB}.$$

Consider now the horizontal forces acting upon the shaded portion D of the same slice as before, which is redrawn on the right of the figure. Let

$$\frac{\text{shaded part of triangle M}}{\text{whole area of triangle M}} = m.$$

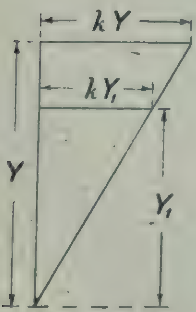

FIG. 68.

Then m (Q — P) is the amount of force acting upon the part D and causing shear along the horizontal section X. Let the intercept R represent m (Q — P). In the same way other intercepts may be formed and their ends joined to give a curve of shearing force. Any intercept of this curve, such as R, represents the horizontal shearing force acting over a length of 1 in. of the cantilever at the same level as the intercept.

That the curve is a parabola is shown thus. Denote as Y the length of the vertical side of the triangle M or N in Fig. 67; let kY in Fig. 68 be the length of the horizontal side. By drawing a line parallel

to the side kY a smaller triangle, similar to the first one, is formed; if Y_1 is the length of its vertical side and k_1Y_1 is the length of its horizontal side,

$$\frac{kY}{Y} = \frac{k_1Y_1}{Y_1}.$$

The equation shows that k is equal to k_1, so that all such horizontal intercepts as k_1Y_1 may be expressed by multiplying the corresponding vertical intercept by a constant number k.

From the area of the whole triangle M (Fig. 67) the area of the unshaded part may be subtracted, giving

$$\text{area of shaded part of triangle} = \frac{kY^2}{2} - \frac{kY_1^2}{2}.$$

The first term of the expression is constant and is represented by the intercept E. In the second term, Y_1 is the distance from the neutral plane of the section where the shearing is considered; this term is represented by the intercept F. The curve through the ends of all such intercepts is therefore a parabola.

At the neutral plane, Y_1 is nothing and therefore $\frac{k}{2}Y^2$ represents the shearing force, equal to $\frac{W}{K}$, at that section. In Fig. 51 the end portion of the beam is identical with a cantilever of rectangular section having an upward instead of a downward acting load. The arm K of the couple resisting the bending moment is shown to be equal to $\frac{2}{3}$ of the depth of this cantilever (see p. 85); if then Y represents half the depth of the cantilever,

$$K = \frac{2}{3} \times 2Y = \frac{4}{3}Y.$$

Hence,

$$\frac{k}{2}Y^2 = \frac{W}{K} = \frac{3W}{4Y}$$

and the constant

$$k = \frac{3W}{2Y^3}.$$

The shearing force over a horizontal sectional area 1 in. long and situated at any level being found, the intensity of the corresponding shearing stress is obtained by dividing the shearing force by the number of inches in the width of the cantilever. Let B represent this number of inches. Then if ordinates such as R are set out to represent values of the expression $\frac{k}{2B}(Y^2 - Y_1^2)$ a curve showing the intensity of the shearing stress at the different horizontal sections

of a cantilever of rectangular section and similar to the shearing force curve will result. Whether the curve is drawn to indicate shearing force or shearing stress intensity it is a parabola when the cantilever or beam—for the relationships discussed apply to beams as well—has a rectangular section with two vertical and two horizontal sides. In the case of the rolled joist (Fig. 58), the shearing force is many times greater in the web than it is in the flanges; the main function of the web is, in fact, to resist the shearing force.

Consider now the shearing forces acting tangentially upon the shaded part H (Fig. 67) which has a length of 1 in. and a depth of t in. The horizontal forces upon the upper and lower surfaces of this slice are nearly equal, but not being exactly equal they do not form a couple. Balance is established in the horizontal direction because the horizontal force represented by the shaded area J,—equal to the difference between the shearing forces acting respectively on the upper and lower surfaces of the slice,—is exerted in opposition to the greater shearing force. The slice being thin, the amount of force represented by the area J is very small; for the present we may leave it out of account and regard the two shearing forces as equal.

Let the width of the cantilever be 1 in. Then if p lb. is the horizontal shearing force in either direction, p lb. per sq. in. will be the shearing stress and pt lb.-in. is the moment of a couple acting on the slice H. If q lb. is the amount of vertical shearing force acting over each end of the slice, q lb. $\times$ 1 in. or q lb.-in. is the moment of a couple formed by the vertical forces. As the slice is at rest under the action of the two couples,

$$q \text{ lb.-in.} = pt \text{ lb.-in.}$$

Hence,

$$p \text{ lb.} = \frac{q}{t} \text{ lb.}$$

Since q lb. is the force spread over an area of t in. $\times$ 1 in., or t sq. in.,

$$\text{vertical shearing stress (lb. per sq. in.)} = \frac{q \text{ lb.}}{t \text{ sq. in.}}$$

But $\frac{q}{t}$ has been shown to equal p, which represents the horizontal shearing stress in lb. per sq. in. Since our assumption that the two horizontal forces are equal is not in strict accordance with facts, it is reasonable to conclude that neither will the last result be either. The thinner the slice is taken, however, the nearer the result will be to the exact truth, and just as the thickness of the slice vanishes to nothing all error disappears. We are quite justified in concluding, therefore, that

The intensity of the vertical shearing stress in a beam or cantilever

at any level is exactly equal to the intensity of the horizontal shearing stress at the same level.

Since the horizontal shearing stress increases from nothing at the part furthest from the neutral plane to a maximum at the neutral plane, it follows that the vertical shearing stress also is nothing at the top or the bottom of a vertical section and attains a maximum at the neutral axis. Taking the depth of a beam or cantilever as 2Y in. (see Fig. 67) and its thickness as B in., the area of a vertical section is 2YB sq. in. If W lb. is the total shearing force distributed over the section,

$$\text{average shearing stress} = \frac{\text{W lb.}}{\text{2YB sq. in.}}$$

On p. 94 the maximum shearing stress (at the neutral surface) is shown to be $\frac{W}{KB}$ lb. per sq. in. On p. 95, $\frac{W}{K}$ is shown to be equal to $\frac{3W}{4Y}$.

Hence,

$$\text{maximum shearing stress} = \frac{W}{KB} = \frac{3W}{4YB}$$

Hence,

$$\frac{\text{average shearing stress}}{\text{maximum shearing stress}} = \frac{W}{2YB} \div \frac{3W}{4YB} = \frac{2}{3}.$$

That is, the average stress is two-thirds of the maximum stress.

A beam of rectangular section, 4 in. wide and 6 in. deep, has a span of 10 *ft. A load of* 2,000 *lb. is supported at a point* 4 *ft. distant from the right-hand support. What is the average intensity of the shearing stress over any vertical section in the part of the beam to the right of the load, and what is the intensity of the shearing stress at the neutral plane in the part to the left of the load?*

R lb. = reaction of right-hand support = $\left\{\begin{array}{l}\text{total shearing force in any}\\ \text{vertical section to the}\\ \text{right of the load.}\end{array}\right.$

Equating turning moments with respect to the left-hand support,

$$10\text{R lb.-ft.} = 2{,}000 \text{ lb.} \times 6 \text{ ft.}$$

$$\text{R} = 1{,}200 \text{ lb.}$$

Area of vertical section = 4 in. × 6 in. = 24 sq. in.

$$\left.\begin{array}{l}\text{Average shearing stress}\\ \text{in a section to the right}\\ \text{of the load}\end{array}\right\} = \frac{1{,}200 \text{ lb.}}{24 \text{ sq. in.}} = 50 \text{ lb. per sq. in.}$$

$$R_1 = \text{reaction of left-hand support} = \begin{cases} \text{total shearing force in any} \\ \quad \text{vertical section to the left} \\ \quad \text{of the load.} \end{cases}$$

$$R_1 = (2{,}000 - 1{,}200) \text{ lb.} = 800 \text{ lb.}$$

$$\left.\begin{array}{l} \text{Shearing force per in. length} \\ \quad \text{of neutral surface to the} \\ \quad \text{left of the load} \end{array}\right\} = \frac{k}{2} Y^2 = \frac{3W}{4Y} = \frac{3 \times 800}{4 \times 3} = 200 \text{ lb.}$$

$$\text{Shearing stress at neutral surface} = \frac{200}{B} = \frac{200}{4} = 50 \text{ lb. per sq. in.}$$

The maximum shearing stress in the second case proves to be equal to the average stress in the first case. The average stress in the second case is ($\frac{2}{3} \times 50$) lb. per sq. in.

Torsion.—A rod or tube, twisted about its axis, is said to be subjected to torsion. The torque applied, together with the angular deflection or amount of twisting, may be measured by the apparatus shown diagrammatically in Fig. 69.

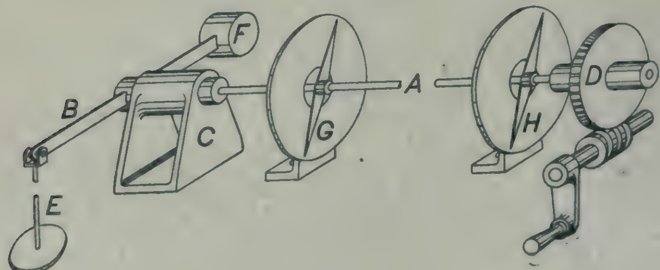

FIG. 69.

The rod A is gripped at each end in such a way that when the worm-wheel D is turned a torque is transmitted through the rod to a spindle which rotates on ball-bearings in the headstock C. An arm B is fixed to the spindle. A load pillar E at one end of the arm B is counterpoised by a load F.

Readings of degrees upon two circular scales are obtained by means of the double pointers G and H. Should the movement of one end of a pointer appear to be more or less than that of the other end, the average of the two observations is found. If each pointer is set to the zero of its scale before a torque is applied, the amount of twisting of the portion of the rod or tube between the pointers, due to any torque to which it is afterward subjected, is indicated by the difference between the average readings of the two scales. The bracket, not shown, which carries the worm and worm-

wheel together with the headstock and the two scales, is clamped to a rigid bedplate not represented in the figure. All of these but the headstock may be moved into different positions. A load being placed on the pillar, the rod is twisted until the lever arm "floats."

The effect of twisting a tube will first be examined. A metal plug was brazed into each end of a mild steel tube having an external diameter of 0·497 in. and an internal diameter of 0·403 in. The thickness was therefore 0·047 in. The distance between the pointers was 30 in. The results obtained by the apparatus were:

Torque (lb.-in.) . . .	12	24	36	48	60	72	84
Deflection (degrees) . . .	0·6	1·2	1·8	2·4	3·1	3·75	4·4

The graph of these numbers is shown in Fig. 70. When the torque

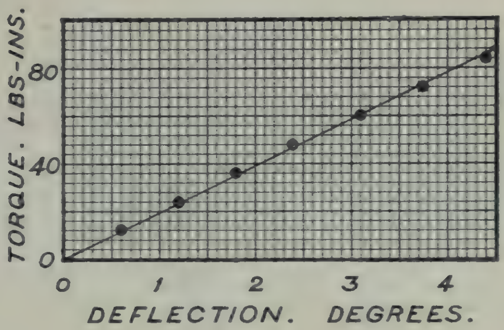

FIG. 70.

was removed, the deflection disappeared which showed that the material was elastic. The slope of the graph is 19·5, and consequently

$$\text{torque (lb.-in.)} = 19{\cdot}5 \times \text{deflection (degrees)}.$$

As one radian contains 57·3 degrees, the torque which produces one radian of angular deflection is $(19{\cdot}5 \times 57{\cdot}3)$ lb.-in. or 1,120 lb.-in.

Is the angular deflection dependent upon the length of tube twisted? The answer to this question is obtained by fixing one of the pointers shown in Fig. 69 at different places on the tube, applying the same torque at each position, and plotting the consequent angular deflection against the length of tube between the two pointers.

Length of tube twisted (in.)	5	10	15	20	25	30
Angular deflection (degrees).	0·75	1·5	2·2	2·9	3·7	4·4

These numbers, obtained by applying a torque of 84 lb.-in., give the straight-line graph (Fig. 71) which shows that *the angular*

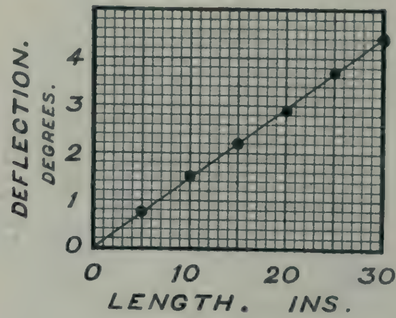

FIG. 71.

deflection produced by a constant torque is proportional to the length twisted.

A straight line drawn along the tube before it is twisted therefore becomes a helix when twisting takes place. Suppose the twisted tube to be developed into a flat sheet, as in Fig. 72, where R represents the mean radius of the tube.

$$R = \left(\frac{1}{2} \times \frac{0{\cdot}497 + 0{\cdot}403}{2}\right) \text{ in.} = 0{\cdot}225 \text{ in.}$$

The dimension D is the development of the curved path of a point in the centre of thickness of the tube at the section where the greatest distortion was measured.

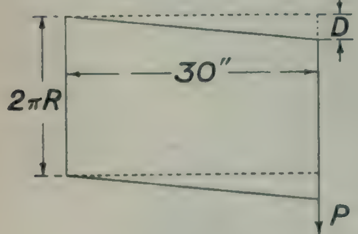

FIG. 72.

Compare the diagram with Fig. 64, which indicates the distortion by shearing of a block of india-rubber. Hence, the tube being 30 in. long and D being in inches, $\frac{D}{30}$ expresses a shear strain which applies not merely to the development but to the actual twisted tube as well. Consequently, the effect of twisting is to produce shearing stress in the tube. The shearing force P lb. causing the distortion

D and distributed over the sectional area of the tube is found by dividing the torque by the mean radius R. The sectional area of the tube is $2\pi Rt$ sq. in., t being the thickness. The area amounts to 0·0665 sq. in. Hence,

$$\left.\begin{array}{l}\text{shearing stress (lb. per}\\ \quad\text{sq. in.)}\end{array}\right\} = \frac{P}{2\pi Rt} = \frac{P \text{ lb.}}{0{\cdot}0665 \text{ sq. in.}}$$

$\frac{D \text{ in.}}{30 \text{ in.}}$ being the shear strain,

$$\text{coefficient of rigidity, } G = \frac{\text{shearing stress}}{\text{shear strain}} = \frac{P \times 30}{0{\cdot}0665 \times D} = 450\,\frac{P}{D}.$$

Let P and D each correspond to a twist of 1 radian. From the graph we have found that a distortion of 1 radian is caused by a torque of 1,120 lb.-in. Hence

$$P = \frac{\text{torque}}{\text{mean radius}} = \frac{1{,}120 \text{ lb.-in.}}{0{\cdot}225 \text{ in.}} = 4{,}970 \text{ lb.}$$

$$D = \text{mean radius} = 0{\cdot}225 \text{ in.}$$

$$\therefore G = \frac{450 \times 4{,}970}{0{\cdot}225} = 9{,}950{,}000 \text{ lb. per sq. in.}$$

The result may be expressed in a general form. Let M lb.-in. of torque be required to produce a deflection of 1 radian, assuming that the limit of elasticity is not exceeded by so large a distortion. If the length in inches of the tube is L,

$$\text{shear strain} = \frac{D}{L} = \frac{R}{L}$$

$$\text{shearing stress} = \frac{\text{torque}}{R} \times \frac{1}{\text{sectional area}} = \frac{M}{2\pi R^2 t}$$

$$G = \frac{\text{shearing stress}}{\text{shear strain}} = \frac{ML}{2\pi R^3 t}.$$

The mean diameter of a tube is 0·5 in.; the thickness of the tube is 0·02 in., and the modulus of rigidity of the material is 8,000,000. One end being fixed, a twist of 6° is caused at the other end by a torque of 2·5 lb.-ft. How long is the tube?

Let L in. be the required length,

$$\text{distance D (Fig. 72)} = \left(\frac{6}{360} \times 2\pi \times \frac{0{\cdot}5}{2}\right) \text{in.} = 0{\cdot}02617 \text{ in.}$$

$$\text{shear strain} = \frac{D}{L} = \frac{0{\cdot}02617}{L}$$

$$\left.\begin{array}{r}\text{shearing force over}\\ \text{the section}\end{array}\right\} = \frac{\text{torque}}{\text{mean radius}} = \frac{2{\cdot}5 \times 12}{0{\cdot}25} = 120 \text{ lb.}$$

$$\text{shearing stress} = \frac{\text{shearing force}}{\text{sectional area}} = \frac{120}{\pi \times 0{\cdot}5 \times 0{\cdot}02} = \left\{\begin{array}{l}3{,}820 \text{ lb.}\\ \text{per sq. in.}\end{array}\right.$$

$$\text{modulus of rigidity} = \frac{\text{shearing stress}}{\text{shear strain}} = \frac{3{,}820 \times L}{0{\cdot}02617} = 8{,}000{,}000$$

$$L = \frac{0{\cdot}02617 \times 8{,}000{,}000}{3{,}820} = 54{\cdot}9 \text{ in. or } 4{\cdot}57 \text{ ft.}$$

What torque, in lb.-ft., will twist through an angle of 3°, a tube of the same thickness and material but with a diameter 1·3, and a length 1·7, that of the tube specified in the preceding example?

Let M be the torque in lb.-in. Referring to Fig. 72 and the preceding results, we have

$$\text{shear strain} = \frac{3 \times 1{\cdot}3 \times 0{\cdot}25}{57{\cdot}3 \times 1{\cdot}7 \times 54{\cdot}9} = 0{\cdot}000183$$

$$\text{shearing stress} = \frac{M}{1{\cdot}3 \times 0{\cdot}25} \div \pi \times 1{\cdot}3 \times 0{\cdot}5 \times 0{\cdot}02 = 75{\cdot}4\,M$$

$$\text{modulus of rigidity} = \frac{75{\cdot}4M}{0{\cdot}000183} = 8{,}000{,}000$$

$$M = \frac{0{\cdot}000183 \times 8{,}000{,}000}{75{\cdot}4} = 19{\cdot}4 \text{ lb.-in.}$$

Expressed in lb.-ft., the required torque is $\frac{19{\cdot}4}{12}$ or 1·617 lb.-ft.

Torsional Strength of Circular Rods.—The results obtained by twisting a tube are of assistance in grasping the problem of the resistance to torsion of a solid rod. Fig. 73 represents the front end of a rod which has been twisted through 1 radian. The radial line which was in the position OA before twisting occupies the position OB after twisting. Regard the rod as built up of thin concentric tubes. Although the front end of each tube has been twisted through 1 radian the shear strain and therefore the shearing stress also are different for each tube.

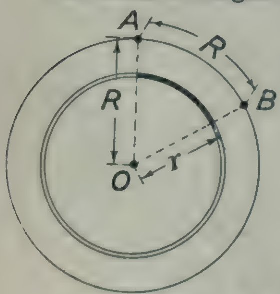

FIG. 73.

Refer to Fig. 72 and it will be evident that whereas the shear strain on the outside of a rod of length L is $\frac{R}{L}$, it is $\frac{r}{L}$ for the imaginary tube having a mean radius r. A rod is subjected to the greatest torque it will stand without injury when the shearing stress at its outer surface has reached the point where plastic yielding begins. If the outside of the rod is strained as far as is practicable, then the cylindrical surface of a core having a radius r is not strained to the extent that would be possible if the core were an independent rod.

The relationship between the diameters of rods and the greatest torque to which each rod may be subjected is deducible from experimental results. A number of silver-steel rods of different diameters but of the same effective length, viz. 30 in., were tested by the apparatus of Fig. 69 in the same way as the tube. Plotting the torque in lb.-in. against radians of twist, graphs like Fig. 70 resulted. The slopes of these graphs, together with the diameters of the rods, are given in the following table :—

Dia. of rod (in.) .	0·125	0·156	0·187	0·219	0·25	0·282	0·312
Slope . .	9·3	22	44·6	86·6	145	246	365

The slope expresses in each case the torque in lb.-in. which would cause a twist of 1 radian. For the largest rod, the shear strain of the outer surface which would be produced by a twist of 1 radian is

$$\frac{0·312}{2} \times \frac{1}{30}, \text{ or } 0·0052.$$

Through what angle must the rod of 0·25 in. diameter be twisted to produce the same extent of shear strain in its outer surface? Look at Fig. 72 and it is clear that the distance D is $\frac{1}{2} \times 0·312$ in. for the outside of the larger rod when that rod is twisted through 1 radian and $\frac{1}{2} \times 0·25$ in. for the outside of the smaller rod when twisted through the same angle. If, however, the outside of the smaller rod is distorted sufficiently to make the dimension D equal to $\frac{0·312}{2}$ in., the corresponding number of radians of twist is $\frac{0·312}{2} \div \frac{0·25}{2}$ or 1·25 nearly and its skin stress is then the same as that of the larger rod. The torque to twist the smaller rod through 1 radian is tabulated as 145 lb.-in. and therefore (1·25 × 145) lb.-in. or 181 lb.-in. (approx.) is the torque required to set up the

same shear stress on the outside of the smaller rod as on the outside of the larger one.

Briefly, if

$$M \text{ (lb.-in.)} = \frac{R}{r} \times T,$$

when R is the radius of the largest rod, r is the radius of any other rod and T is the torque in lb.-in. to twist the rod of radius r through 1 radian, then M lb.-in. is the torque which causes the same shear stress in the outer skin of the rod of radius r as is produced in the skin of the rod of radius R by twisting it through 1 radian.

Calculating M in this way for each rod, we have the following table :—

Dia. of rod (in.) .	0·125	0·156	0·187	0·219	0·25	0·282	0·312
M (lb.-in.) . .	23·2	44	74·5	123	181	272	365

When plotted, these numbers give the graph (Fig. 74). The interpretation of a graph in which the numbers along one axis vary

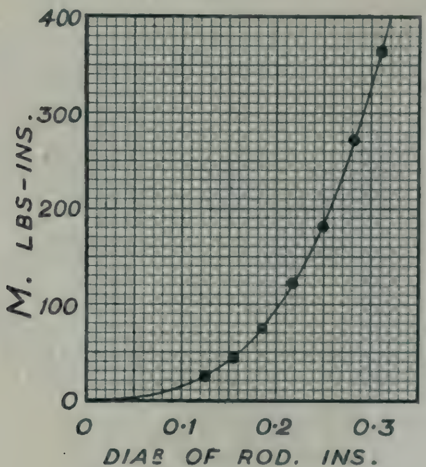

Fig. 74.

according to the squares of the corresponding numbers along the other axis is given on p. 16. Plotting the values of M against the squares of the diameters of the rods results, however, in another curved graph of such a nature as to suggest that the torque M may vary according to a higher power of the diameter than 2. A table of the cubes of the diameters, together with the torques, is given.

Dia. of rod (in.) .	0·125	0·156	0·187	0·219	0·25	0·282	0·312
(Dia. of rod)³ .	0·00195	0·0038	0·00655	0·0105	0·0156	0·0224	0·0304
M (lb.-in.) . .	23·2	44	74·5	123	181	272	365

Plotting the last two columns, Fig. 75 is obtained. Repre-

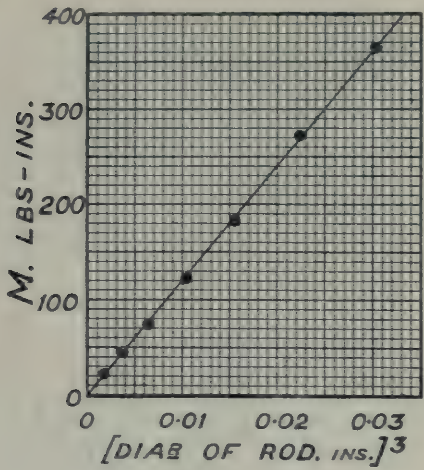

FIG. 75.

senting the diameter of the rod in inches by D, the equation to the graph is

$$M \text{ lb.-in.} = 11{,}900 \ D^3.$$

The equation expresses the fact that if a number of rods of equal length but of different diameters are twisted until they all attain the same shear strain—and consequently shearing stress also—at their outer skins, the torques required will vary as the cubes of the diameters of the rods. Briefly expressed : *The torsional strengths of circular rods of the same elastic material vary as the cubes of the rod diameters.*

Experimental data is not essential in order to obtain an expression for the torsional strength of a circular rod. Regard a rod of radius R as built up of thin, closely fitting, concentric tubes. Let r_1 be the internal, r_2 the external, and r the mean radius of one such tube. Refer to Fig. 73 and it will be evident that the ratio of the shear strain of this tube to the shear strain of the outer surface of the tube is r : R, for the length of the tube equals that of the rod. If f denotes the intensity of shearing stress in the skin of the rod and

f_1 the shearing stress intensity in the tube, then since the stress is proportional to the strain,

$$f_1 : f :: r : R, \text{ or } f_1 = \frac{r}{R} f.$$

The shearing force distributed over the section of the tube is the product of the shearing stress intensity and the sectional area; that is

$$\text{shearing force} = \frac{r}{R} f \times \pi (r_2^2 - r_1^2).$$

The torque exerted upon the tube is equal to the moment about the axis of the tube of the shearing force distributed over the section. The mean distance from the axis at which every part of this force acts is r and hence,

$$\text{torque exerted upon the tube} = \frac{r^2}{R} f \times \pi (r_2^2 - r_1^2).$$

Let the values of the factor $\frac{r^2}{R} f$ for a series of closely fitting tubes be plotted vertically as in Fig. 76. Horizontally plot πr_1^2, and πr_2^2. These values being represented by the lengths A and B respectively, the difference between A and B is the sectional area of the tube. The length C represents πr^2, which has a value midway between those of πr_1^2 and πr_2^2, if the difference between r_1 and r_2 is exceedingly small. The ordinate D represents $\frac{r^2}{R} f$. If f is the greatest stress to which the material may be subjected, the shaded area represents the strength of the tube; that is, the torque it is capable of resisting. If ordinates similar to D are formed for a number of other tubes, the unshaded rectangular areas will represent the strengths of these tubes. Through the upper ends of the ordinates a graph is drawn which is straight, because the ratio of any pair of co-ordinate values is constant. Thus

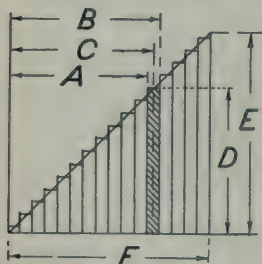

FIG. 76.

$$\text{slope of graph} = \frac{r^2}{R} f \div \pi r^2 = \frac{r^2 f}{R \pi r^2} = \frac{f}{\pi R}.$$

The result is a constant, because neither f nor R changes as we deal successively with the tubes.

The triangular area under the graph is equal to the sum of the rectangular areas; since this sum represents the total torque

exerted upon the rod, the triangular area expresses the total torque. The ordinate E represents a factor of the torque acting on the skin, of the value $\frac{R^2}{R}f$, for at the outside, r becomes equal to R; the intercept F represents πR^2 and

$$\text{area of triangle} = \text{total torque} = \frac{1}{2} \times \frac{R^2}{R} f \times \pi R^2 = \frac{\pi}{2} f R^3.$$

If the diameter is represented by D,

$$R = \frac{D}{2}$$

Substituting in the preceding equation

$$\text{total torque} = \frac{\pi}{2} f \left(\frac{D}{2}\right)^3 = \frac{\pi}{16} f D^3.$$

Refer now to p. 105, where the experimental result is given, viz.,

$$\text{total torque (M lb.-in.)} = 11{,}900\ D^3.$$

The numerical coefficient of D^3 is the value of $\frac{\pi}{16} f$; if, therefore, f is the common skin stress attained by the rods when each is subjected to a torque of M lb.-in. as tabulated on p. 104,

$$f = \frac{16 \times 11{,}900}{\pi} = 60{,}600 \text{ lb. per sq. in.}$$

It is not suggested that the rods were twisted until this stress was actually reached, for the amount of twisting was supposititious and chosen for convenience.

The safe limit of stress at the outside of a circular rod, 1·4 *in. in diameter, is reached when a torque of* 5,000 *lb.-in. is applied. What is the maximum torque that may safely be applied to a rod of the same length and of similar material,* 2·2 *in. in diameter?*

$$M = \text{required torque in lb.-in.}$$

Since the rod diameter is the only dimension that is altered,

$$\frac{\text{strength of 1st rod}}{\text{strength of 2nd rod}} = \frac{(1{\cdot}4)^3}{(2{\cdot}2)^3} = \frac{5{,}000}{M}$$

$$M = \frac{10{\cdot}65 \times 5{,}000}{2{\cdot}74} = 19{,}400 \text{ lb.-in.}$$

A torque of 840 *lb.-ft. is applied to a circular rod having a diameter of* 1·8 *in. What is the shearing stress in the outer skin of the rod?*

$$\text{Torque (lb.-in.)} = \frac{\pi}{16} \times \text{shearing stress} \times (\text{diameter})^3.$$

$$\text{Shearing stress} = \frac{16}{\pi} \times \frac{840 \times 12}{(1{\cdot}8)^3} = 8{,}800 \text{ lb. per sq. in.}$$

Strength of Hollow Shafts.—If R in. is the external radius and R_1 in. is the internal radius of a hollow shaft, its strength is that of a solid shaft of radius R in. less the strength of a shaft of radius R_1 in. If f lb. per sq. in. is the external skin stress and f_1 lb. per sq. in. is the internal skin stress,

$$f_1 = \frac{R_1}{R} f.$$

As f_1 is the skin stress of the missing core, the strength of the hollow shaft is expressed as follows:

Strength of solid shaft − strength of core.

$$\frac{\pi}{2} f R^3 - \frac{\pi R_1}{2R} f R_1^3.$$

By subtraction,

$$\begin{array}{c}\text{strength of hollow shaft} \\ \text{(lb.-in. of torque)}\end{array} = \frac{\pi}{2} f \left(R^3 - \frac{R_1^4}{R}\right)$$

$$= \frac{\pi}{2} f \left(\frac{R^4 - R_1^4}{R}\right)$$

Let D be the external and D_1 the internal diameter; then

$$R = \frac{D}{2}$$

$$R_1 = \frac{D_1}{2}$$

By substitution,

$$\begin{array}{c}\text{strength of hollow shaft} \\ \text{(lb.-in. of torque)}\end{array} = \frac{\pi}{2} f \left(\frac{\frac{D^4}{16} - \frac{D_1^4}{16}}{\frac{D}{2}}\right) = \frac{\pi}{16} f \left(\frac{D^4 - D_1^4}{D}\right)$$

The Modulus of Rigidity for a Circular Rod.—The equation

$$M = \frac{\pi}{2} f R^3$$

in which f represents the skin stress in a circular rod of radius R produced by a torque of M lb.-in., enables us to find the modulus of rigidity G for the material of the rod, provided that we know also the length, L in., of the rod and the angle, θ radians, through which it is twisted. From the last equation,

$$\text{shearing stress in the skin} = f = \frac{2M}{\pi R^3}.$$

The effect of the twisting is to cause a movement θR, along a circular path of radius R, of a point on the outside of the rod at its twisted end. θR corresponds to the dimension D in Fig. 72. Hence,

$$\text{shear strain} = \frac{\theta R}{L}$$

$$\text{modulus of rigidity} = G = \frac{\text{shearing stress}}{\text{shear strain}} = \frac{2M}{\pi R^3} \div \frac{\theta R}{L} = \frac{2ML}{\pi \theta R^4}.$$

The table on p. 103 shows that a torque of 145 lb.-in. was required to twist a silver-steel rod of 0·25 in. diameter and 30 in. long through an angle of 1 radian. The modulus of rigidity for the material of the rod is therefore,

$$G = \frac{2 \times 145 \times 30}{\pi \times 1 \times (0 \cdot 125)^4} = 11{,}300{,}000 \text{ lb. per sq. in.}$$

EXAMPLES.

65. A beam of uniform section is 10 ft. long, is supported at each end and weighs 1,000 lb. What is the bending moment in lb.-ft. at a vertical section 2 ft. from one of the supports?
66. Two beams, A and B, of rectangular section and of similar material, are of equal length. The breadth and depth respectively of A are 4 in. and 6 in.; those of B are 3 in. and 8 in. What is the ratio of the strength of A to that of B?
67. Compare the strengths of two beams, each loaded at the centre, and each of rectangular section, 6 in. by 3 in.; the first one has the larger side vertical and is of 20 ft. span, the second has the shorter side vertical and is of 10 ft. span.
68. If a beam 12 ft. long and 5 in. square section is just strong enough to carry 1 ton at its centre, what weight may be placed at the centre of a beam of the same material 20 ft. long, 6 in. broad, and 11 in. deep?

69. If a rectangular beam 20 in. long, 1 in. wide, and 1 in. deep, supported at each end, will just support 350 lb. at its centre, what should be the depth of a beam of the same material 60 in. long and 3 in. wide in order to support at the centre a load of 2,000 lb.?
70. A strip of wood is supported horizontally at points 40 in. apart; a load of 30 lb. is placed at the centre. If instead of this load two loads each weighing W lb. are placed on the beam, one at 10 in. and the other at 15 in. from one of the supports, with the result that the same bending moment as before occurs at the centre, how much is W?
71. A wooden beam of rectangular section is just strong enough to support at the centre a load of 4,000 lb. The length, breadth, and depth respectively are 5 ft., $3\frac{1}{8}$ in., and 4 in. What load will a beam of similar material carry if its length, breadth, and depth are respectively 10 ft., 5 in., and 8 in. when the load is placed at a distance of 2 ft. from one of the supports?
72. A girder of 40 ft. span carries a load of 5,000 lb. per ft. run. Calculate the bending moment at the centre and assuming that the flanges take all bending moment, determine their cross-sectional areas. The allowable stress in the metal may be taken as 10,000 lb. per sq. in., and the mean distance apart of the flanges as 54 in.
73. A girder, supported at each end, has a span of 22 ft. At a distance of 2 ft. from one end a load of 10 tons is placed. At distances of 7 ft. and 13 ft., respectively, from the first load, loads of 6 tons and 7 tons are placed on the girder. Find in tons-ft. the bending moment at each section where a load is situated and draw a diagram of bending moment.
74. A slab of indiarubber, 5 in. square and 1 in. thick, has one square face cemented to a vertical fixed surface. If the Modulus of Rigidity of the rubber is 60 lb. per sq. in., what tangential load, distributed evenly over the free vertical surface of the slab, will cause that surface to move downward to the extent of 0·1 in.?
75. A beam 20 ft. long, supported at both ends, carries three loads of 10 tons each. Calling one end of the beam A, the respective distances of the loads from A are 4 ft., 9 ft., and 16 ft. Determine the shearing force in tons and the bending moment in tons-ft. at a section 15 ft. from A.
76. A beam 10 ft. long rests on two supports which are respectively 1 ft. and 7 ft. from the left-hand end of the beam. A weight of 10 lb. hangs from the left-hand end of the beam, and a weight of 5 lb. from the right-hand end. Find the two reactions and also the bending moment and shearing force at the centre of the beam.
77. A girder, which rests on supports at its ends, is 10 ft. long, and carries a uniformly distributed load of 1 ton per ft. run over

5 ft. of its length, the load commencing $1\frac{1}{2}$ ft. from the left-hand support. Calculate the shearing force in tons and the bending moment at a distance of 3 ft. from the left-hand support.

78. A plate girder is constructed by riveting a single vertical web to horizontal flanges. The depth of the web is 20 in. and the span of the girder is 20 ft. Including the weight of the girder, the total load, which is evenly distributed, is 42 tons. Assuming that the average shearing stress may be 3 tons per sq. in., how thick should the web be at a section in the neighbourhood of one of the supports?

79. A tube, 3 ft. long, has a mean diameter of 1·5 in. and the material is 0·025 in. thick. If one end is fixed and a torque is applied to the other, what amount of torque, in lb.-ft., will cause the free end of the tube to twist through 4°? The Modulus of Rigidity is 9,000,000 lb. per sq. in.

80. The safe limit of stress at the outside of a circular rod, 1·4 in. in diameter, is reached when a torque of 5,000 lb.-in. is applied. What is the maximum torque that may safely be applied to a rod of similar material, 2·2 in. in diameter?

81. A torque of 840 lb.-ft. is applied to a circular rod having a diameter of 1·8 in. What is the shearing stress in the outer skin of the rod?

82. Find the torque, in lb.-ft., necessary to produce a skin stress of 12,000 lb. per sq. in. in a circular rod 1·5 in. in diameter.

83. Assume that the propeller shaft of a ship is subjected to pure torsion and that the safe shearing stress is 5 tons per sq. in. If the engine develops 10,000 H.P. at a speed of 120 revs. per min., what diameter should the shaft have?

84. The propeller shaft of a vessel having engines which develop 1,000 H.P. at 60 revs. per min., has a diameter of 8 in. Assuming the shaft to be subjected to pure torsion, and that the maximum twisting moment on the shaft is $1\frac{1}{4}$ times the mean, estimate the maximum shearing stress induced in the shaft.

85. The screw shaft of a marine engine is 10 in. in diameter, and the revs. 100 per min. It is replaced by twin screw shafts rotating 150 times a min. If the total H.P. developed in the two cases is the same, and the working stress also is the same in the twin screw shafts as in the single screw shaft, find the proper diameter of shafts in the second case, and compare their weights to that of the shaft replaced.

CHAPTER V

Measurement of Energy.—The capacity of a machine is measured by its rate of output of energy; if the efficiency of the machine is to be determined the rate of input must be measured as well as that of output. In obtaining the output it is usually convenient to employ a form of apparatus in which the energy, after measurement, is expended in overcoming the frictional resistance of some part of the apparatus itself. Brake dynamometers (see "Applied Mechanics—First Year") are used for this purpose; the testing of a pump affords an example of a similar method.

Pump Horse-Power.—In Fig. 77 a rotary pump A, driven by a belt, draws water from a tank through a suction pipe B and discharges it by means of the pipe CD into a vertical vessel E which is several feet high. The lower end of this vessel is closed by a plate containing one or more circular holes through which the water flows back into the suction tank. The lower end of pipe D terminates in a cone formed of thin perforated sheet metal so that there may be the least possible amount of resultant eddying motion in the water. The height of the water-level in E is shown by means of a glass tube F. The rate at which the water flows through the hole or holes in the plate is indicated by a scale at the side of the glass tube.

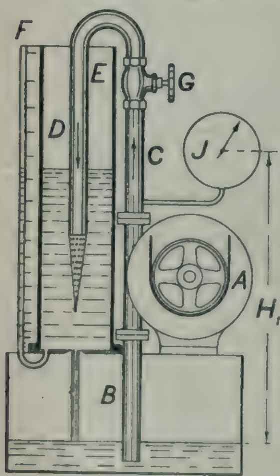

FIG. 77.

The resistance to be overcome by a pump is usually due to the weight of a column of water the height of which is equal to the difference between the water-levels of the suction and the delivery tanks. Partly to avoid the inconvenience of lifting the water through a considerable height, and partly for the sake of easily increasing or decreasing the resistance which the water offers to being forced through the pump, a valve G is placed in the delivery pipe when the only object aimed at is the testing of the pump. Should the valve be nearly closed the resistance of the water is great; the more the valve is opened the less the resistance becomes.

When water passes under the valve the speed and consequently the K.E. of the water also are increased because the sectional area

of the channel is reduced. The rise in pressure of the water approaching the valve is rendered necessary in order to provide the additional K.E. After passing the valve the speed is reduced, and as this is effected the extra K.E. is dissipated by the friction caused by an eddying motion of the water. A pressure-gauge J connected to the delivery pipe is used to measure the resistance of the water.

Since the weight of a column of water 1 ft. high and 1 sq. in. in section is about 0·433 lb., the weight in lb. of a column of the same section and H ft. high is 0·433H. If a pressure-gauge connected to the base of this column indicates a pressure of P lb. per sq. in., then

$$P = 0{\cdot}433H$$

or

$$H = \frac{P}{0{\cdot}433}.$$

So far as the energy expended in forcing water through the pump is concerned, it is indifferent whether the resistance is due to the actual weight of a column of water or the pressure induced by the partial closing of the valve. Hence the energy expended in forcing W lb. of water through a pipe where it is at a pressure of P lb. per sq. in. is the same as that expended in lifting W lb. through a height of H ft. when $P = 0{\cdot}433H$.

The pressure P applies to the level at which the centre of the gauge is situated, but as the water before attaining this level must previously be raised through a height H_1 ft. (see diagram)—which is independent of any pressure indicated by the gauge—the total virtual lift of the water is $(H + H_1)$ ft. The energy output in raising W lb. of water per min. is therefore $W(H + H_1)$ ft.-lb. It is usual to express the rate of energy output in horse-power, and to term it *pump horse-power*. Thus,

$$\text{pump horse-power} = \frac{W(H + H_1)\text{ ft.-lb.}}{33{,}000}.$$

An experiment conducted with an arrangement exactly as shown in Fig. 77 yielded the following results.

$$\text{Flow of water} = 230 \text{ lb. per min.}$$

$$\text{Pressure-gauge reading} = 20 \text{ lb. per sq. in.}$$

$$\text{Height } H_1 = 4 \text{ ft.}$$

$$H = \frac{20}{0{\cdot}433} = 46{\cdot}2 \text{ ft. nearly}$$

$$\text{Total head} = (46{\cdot}2 + 4)\text{ ft.} = 50{\cdot}2 \text{ ft.}$$

$$\text{Water horse-power} = \frac{230 \times 50{\cdot}2}{33{,}000} = 0{\cdot}35.$$

Indicated Horse-Power.—The energy input of an engine, operated by means of oil, gas, or steam, is measured by the use of an arrangement represented in a simplified form in Fig. 78. A small cylinder A forming part of an instrument known as an *indicator* contains a piston which when forced upward compresses a steel spring. A rod connected to the piston carries a pencil B. The spring is so proportioned that a given intensity of steam pressure upon the lower side of the piston is required to raise the pencil 1 in. If this intensity is 120 lb. per sq. in., say, the number 120 is called the *spring scale*. When in use, the cylinder of the indicator is connected by a pipe to the engine cylinder; by means of a three-way cock, communication may be established between the indicator and the steam acting upon either side of the engine piston.

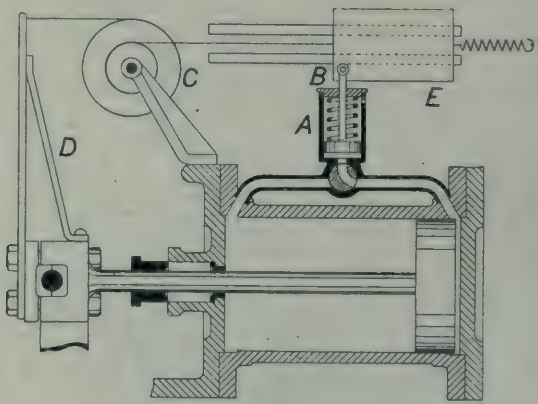

FIG. 78.

At any moment, therefore, the intensity of pressure upon the engine piston is indicated by the height of the pencil B, for the amount of compression of the spring is proportional to the upward pressure upon the indicator piston.

A small two-stepped pulley C turns upon a pin fixed in a bracket. A cord is wound around each step and fastened to the pulley. The cord on the larger step is fixed to a bracket D projecting from the crosshead. The other cord is fastened to a small board E, which slides along a guide frame. A spring continually pulls the board to the right.

The motion of the board E which occurs when the contrivance is in action is controlled by the crosshead. The length of the path of the board E bears the same ratio to the piston-stroke as the diameter of the smaller step to that of the larger one, and at any given instant both the engine piston and the board E will have completed the same fraction of the lengths of their respective paths.

Any such arrangement as this, used to obtain a reduced copy of the motion of a piston, is termed a *reducing gear*.

As the board E has a lateral motion whilst the pencil moves vertically in response to any change of pressure, the result is that a curve called an *indicator diagram* (Fig. 79) is traced upon a sheet of paper pinned to the board. The three-way cock being used to change the connection of the indicator to the other end of the cylinder, a second diagram is traced. The one formed when the steam presses upon the side of the engine piston remote from the crank is called the *head-end diagram;* the other is termed the *crank-end diagram.*

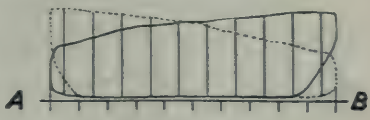

FIG. 79.

Dividing the area enclosed by each diagram into a number of vertical strips of equal width, finding and adding together the heights—measured at the centre of each—of the strips, the average height of the diagram is obtained by dividing the sum of these heights by the number of strips. The vertical direction is determined by a horizontal line AB traced on the paper when no pressure is acting upon the indicator piston. The resulting average height is multiplied by the spring scale to obtain the average *intensity* of pressure upon the engine piston.

Multiplying the number of sq. in. contained by the area of the engine piston into the average intensity of pressure gives the total average force acting upon the piston throughout the length of the stroke. The product of the number of feet per min. moved by the engine piston and the total average force in lb. acting upon it gives the number of ft.-lb. of energy imparted to the engine by the steam (or other fluid) during an interval of 1 min. The rate at which this energy is supplied is termed *indicated horse-power.*

So elementary a form of indicator as the one described became obsolete many years ago. An indicator of modern type is shown in Fig. 80. In this instrument the pencil B is carried at the end of a light steel arm connected to the rod of the piston by a link A. By this means a considerable pencil movement is secured for a small degree of compression of the spring. The sliding board is replaced by a cylindrical drum C to which the paper is attached by clips. A spring inside the drum keeps taut the cord D which is operated by the movement of the crosshead and causes rotation of the drum.

In finding the indicated horse-power (or I.H.P.) of a steam-engine, crank-end and head-end diagrams are required, because the valves may be so set that the average intensity of pressure at one end of the engine cylinder differs from that at the other. In estimating the work done on the crank-end side of the piston allowance must be made for the fact that the area of the piston subjected to pressure is reduced by the amount of the sectional area

of the piston-rod. Particulars of an experiment upon a small horizontal steam engine follow.

Diameter of piston = 4 in.
Length of stroke = 0·5 ft.
Diameter of piston-rod = 0·75 in.
Revs. per min. = 210.

From these data we have

Area of piston = 12·566 sq. in.
Area of piston-rod section = 0·442 sq. in.
Effective piston-area at crank-end = (12·566 — 0·442) sq. in.
= 12·124 sq. in.

The diagrams obtained were those reproduced to a half-size scale

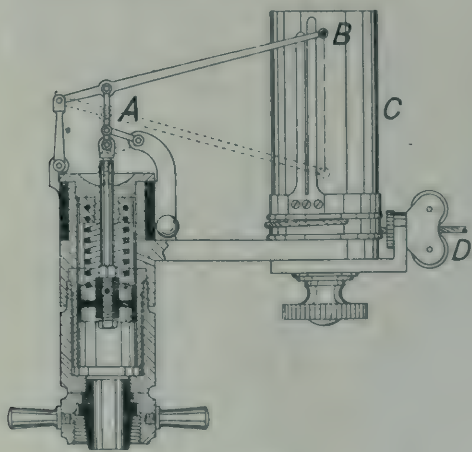

Fig. 80.

in Fig. 79, the dotted one being from the crank-end. The scale of the indicator spring was 40.

Mean height of crank-end diagram = 0·718 in.
Mean height of head-end diagram = 0·728 in.

Mean effective pressure at crank-end = $0·718 \times 40 = 28·72$ lb. per sq. in.

Mean effective pressure at head-end = $0·728 \times 40 = 29·12$ lb. per sq. in.

Total pressure at crank-end = $(28·72 \times 12·124)$ lb. = 348 lb.

Total pressure at head-end = $(29·12 \times 12·566)$ lb. = 366 lb.

In one stroke the force at either the crank-end or head-end is exerted through 0·5 ft., and as the crank shaft turned 210 times in 1 min. each force was exerted through a distance of $(210 \times 0{\cdot}5)$ ft. or 105 ft. This is equivalent to the exertion of a force of $(348 + 366)$ lb. or 714 lb. through 105 ft. in each min. Consequently,

$$\text{I.H.P.} = \frac{105 \times 714}{33{,}000} = 2{\cdot}27.$$

Transmission Dynamometers.—In determining the supply of energy to a machine which is not a prime-mover, some contrivance is required to measure the flow of energy from its source to the machine which it actuates. An apparatus constructed for this

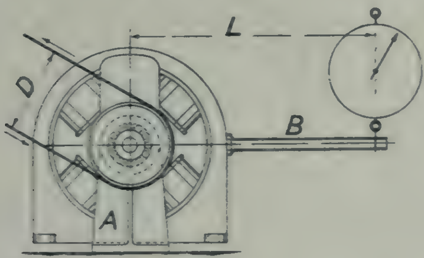

FIG. 81.

purpose is called a transmission dynamometer; its function resembles that of a gas-meter.

In the case of electrical energy the dynamometer may be made as shown in Fig. 81. An electric motor instead of being bolted down to a floor is supported by two brackets, one of which, A, is shown. The base of the motor is raised slightly above the floor-level. The connection between the motor and the supporting brackets is made by a pair of ball-bearings which are concentric with the armature spindle.

When the motor is at work the field-magnets cause a torque to be exerted upon the armature spindle which carries the driving pulley shown in the figure. An exactly equal contrary torque is exerted by the armature upon the frame to which the field-magnets are attached. The torque is measured by means of an arm B, fixed to the frame and prevented from turning round by a pull which is measured by a spring-balance. If this pull is P lb. and the dimension L in the diagram is in ft., then the torque is $P \times L$ lb.-ft. If the rotational speed of the armature is N revs. per min. the number of radians turned per min. is $2\pi N$. Hence,

$$\text{transmitted H.P.} = \frac{PL \times 2\pi N}{33{,}000}.$$

It is preferable, however, to measure the speed of the pulley to which the belt is carried, as there may be some slipping of the belt; the driven machine would then be debited with more energy than it received if the speed of the armature were considered. Let D denote the diameter of the motor-pulley and D_1 that of the pulley

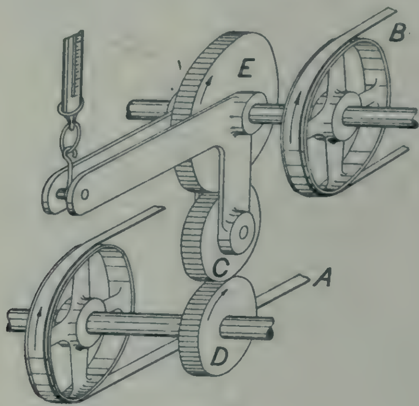

FIG. 82.

on the driven machine. Let E be the effective belt tension. Then,

$$\text{torque on armature spindle} = PL = E \times \frac{D}{2}.$$

From this

$$E = \frac{2PL}{D}$$

$$\text{torque on shaft of driven machine} = E \times \frac{D_1}{2}.$$

Replacing the symbol E by the alternative expression,

$$\text{torque on shaft of driven machine} = \frac{D_1}{D} \times PL.$$

If N represents the number of revs. per min. of the driven pulley,

$$\text{transmitted H.P.} = \frac{PL \times 2\pi N}{33{,}000} \times \frac{D_1}{D}.$$

The essential parts of another form of transmission dynamometer are represented in Fig. 82. Energy is transmitted to the apparatus by a belt A, and from the apparatus to the machine which receives it by another belt B. A bell-crank lever, free to turn around the upper shaft, carries at the end of its vertical arm a spur-wheel C which is driven by another spur-wheel D keyed to the lower shaft,

The wheel C transmits its motion to a third wheel E keyed to the upper shaft.

The forces which tend to cause rotation of the bell-crank lever are represented in Fig. 83. P lb. is the pressure exerted upon the teeth of the wheel C by the teeth of the wheel D. All the energy received at the lower part of this wheel is transferred, at the same rate as it is received, to the wheel E. The force exerted *by* the upper teeth of the wheel C acts toward the *left*, but P_1 indicated in Fig. 83 is a force equal to P, and is the reaction of the teeth of E *upon* the teeth of C. P_1 is equal to P, and the total force with which the wheel C presses against the pin O, around which it turns, is $P + P_1$ or 2P.

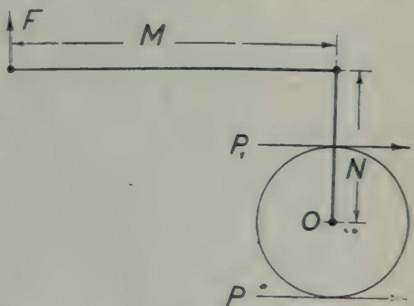

FIG. 83.

An upward force F measured by a spring-balance is exerted at the end of the horizontal arm in order to prevent the bell-crank lever from turning. Equating the turning effects exerted on the bell-crank lever,

$$2PN = FM,$$

from which,

$$P = \frac{FM}{2N}.$$

The dimensions M and N are indicated in Fig. 83.

As all the quantities to the right of the equation may be found by observation, P lb., the pressure on either the upper or the lower teeth of the wheel C, results. Since P lb. is the amount of reaction exerted by the teeth of E, it is also the measure of the pressure exerted *upon* those teeth. If R ft. represents the radius of the pitch circle of the wheel E in Fig. 82, PR lb.-ft. is the torque exerted upon the upper shaft. When the speed of rotation of the upper shaft is known, no further data is required for calculating the transmitted H.P.

Fig. 84 shows another form of transmission dynamometer. Energy is supplied by means of a belt A, and by means of another belt B is transmitted to the machine in which it is utilised. The pulley C is fixed to a sleeve which turns around the shaft. A mitre wheel D is also fixed to the sleeve. Two mitre wheels, E and F, each free to turn about a gudgeon pin, are operated by the wheel D. A mitre wheel G keyed to the shaft is turned by the concurrent action of the wheels E and F. The pulley H is keyed

to the shaft which is driven by the wheel G. The gudgeon pins, about which the wheels E and F rotate, form part of a lever-arm which is prevented from turning about the shaft by a pull measured by a spring-balance.

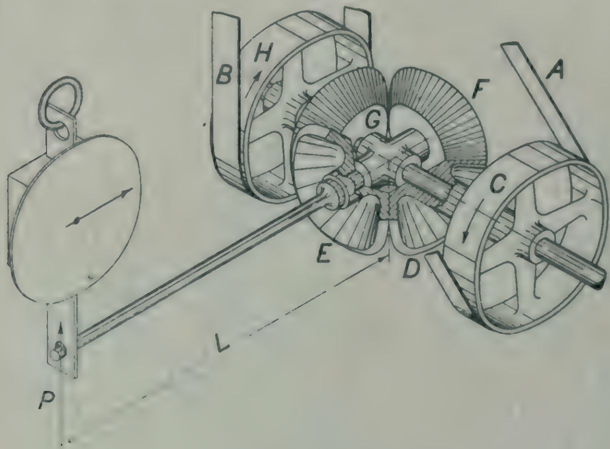

FIG. 84.

Fig. 85 is a diagram of the forces in action. Observing the direction of rotation of the pulley C as given in Fig. 84 we see that a downward pressure P_1 is exerted upon the teeth of the wheel E. A downward reaction P_2 of the teeth of G is also exerted upon the

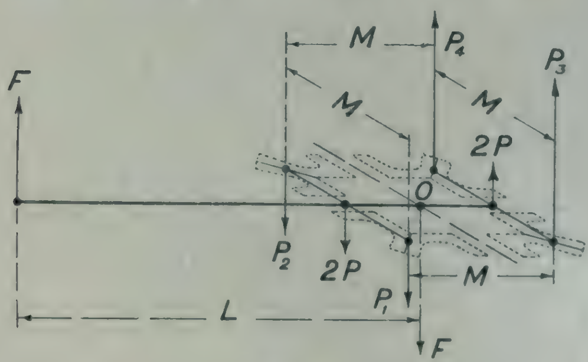

FIG. 85.

wheel E. An upward force P_3, due to the wheel D, is exerted upon F, and an upward reaction P_4 is also exerted upon F.

If the mitre wheels are formed with perfect accuracy the forces P_1 P_2 P_3 P_4 are all of the same amount and act through the corners of a horizontal square of which the length of the side is M. The

centre O is the intersection of the axes of the arm and the shaft. Let P lb. represent each of the four vertical forces. Then P_1 and P_2 result in a downward pressure equal to 2P upon the horizontal arm, whilst P_3 and P_4 result in an upward pressure upon it of the same amount.

The two resultant pressures upon the arm form a couple of which the moment is $2P \times M$. The tendency of the couple to rotate the arm about the shaft is resisted by another couple of which the moment is $F \times L$ when F is the pull in the spring-balance, the

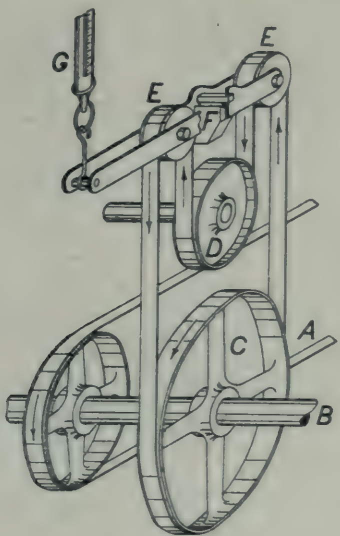

FIG. 86.

dimension L being indicated in the figure. A downward force of F lb., due to the reaction of the shaft to the pressure of the arm, is one of the pair of forces forming the second couple. When equilibrium exists,

$$2PM = FL.$$

The torque exerted upon the wheel G, and therefore upon the shaft also, is due to the couple PM. From the preceding equation,

$$PM = \frac{FL}{2}.$$

There is no occasion to measure either P or M, for F and L may be readily obtained.

If either of the wheels E and F were removed an increase of pressure between the teeth, the setting up of a pressure between the sleeve and the shaft and of a pressure between the shaft and its

bearings would result. These effects it is desirable to avoid, as the consequent frictional resistance would affect the reading of the spring-balance ; otherwise the action of the apparatus would not be disturbed, for apart from the friction effect, FL will still be equal to twice the torque exerted on the shaft.

If the shaft rotates at N revs. per min. we have,

transmitted H.P. = { torque applied to the shaft (lb.-ft.) × rotational speed of shaft in radians per min. ÷ 33,000,

or

$$\text{transmitted H.P.} = \frac{FL}{2} \times \frac{2\pi N}{33{,}000}.$$

A "*Tatham*" *transmission dynamometer* is represented in Fig. 86. A belt A actuated by a prime mover drives the shaft B. A pulley C is keyed to the shaft B. D is the driving pulley of a machine to which the energy is transmitted. Two guide pulleys EE, running on ball-bearings, rotate about pins which are attached to a lever. The lever turns about a knife-edge resting on a bracket F.

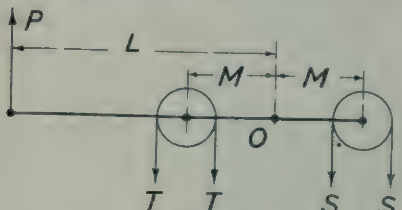

FIG. 87.

To establish equilibrium in the lever a pull, measured by a spring-balance G, is required. An endless belt connects the pulleys C D E E.

The forces acting upon the lever are indicated in Fig. 87. T lb. represents the tension in the tight part, and S lb. the tension in the slack part of the belt. It is evident that the tight part of the belt causes one guide pulley to exert a pressure of 2T lb. upon its spindle, a force which is transmitted to the lever. Similarly there is a pressure of 2S lb. upon the spindle of the other guide pulley. P lb. is the upward pull exerted through the spring-balance.

Using the dimensions indicated in Fig. 87 all the tendencies to induce the lever to turn about the knife-edge at O are balanced when

$$2TM = 2SM + PL.$$

Hence,

$$2M(T - S) = PL,$$

and

$$T - S = \frac{PL}{2M}.$$

All the quantities on the right-hand being easily obtainable, T — S, the effective belt tension, at once follows. The diameter and speed of rotation of the pulley D being found, nothing further is required in order to calculate the transmitted H.P

A dynamometer of the Tatham type being used to carry out a test on the rotary pump illustrated in Fig. 77 the following results were obtained :—

$$\begin{aligned}
\text{Radius of pulley on rotary pump (R)} &= 0{\cdot}333 \text{ ft.}\\
\text{Length of lever (L)} &= 1{\cdot}5 \text{ ft.}\\
\text{Distance M} &= 0{\cdot}5 \text{ ft.}\\
\text{Pull in spring-balance (P)} &= 12 \text{ lb.}\\
\text{Speed of pump pulley (N)} &= 510 \text{ rev. per min.}\\
\text{Transmitted H.P.} &= \frac{PL}{2M} \times \frac{2\pi NR}{33{,}000}\\
&= \frac{12 \times 1{\cdot}5 \times 2\pi \times 510 \times 0{\cdot}333}{1 \times 33{,}000}\\
&= 0{\cdot}583.
\end{aligned}$$

The observations for the last experiment were made concurrently with those given on p. 113. Since the water H.P. measures the rate of energy output and the transmitted H.P. measures the rate of energy input,

$$\text{mechanical efficiency of the pump} = \frac{\text{W.H.P.}}{\text{transmitted H.P.}} = \frac{0{\cdot}35}{0{\cdot}583} = 0{\cdot}6.$$

Transmission of Energy by Belts.—Fig. 88 shows two spring-balances A and B. The latter is supported from a screwed rod provided with a milled nut so that it may be raised or lowered. Encircling half the circumference of a belt-pulley C there is a thin flexible band which in obtaining the following experimental results consisted of a thin woven material.

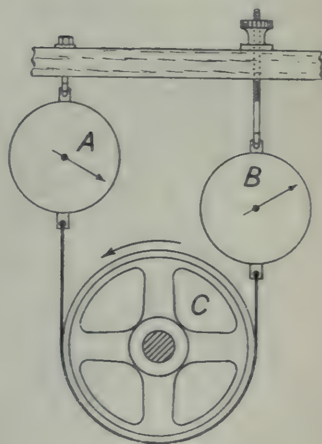

FIG. 88.

When the pulley was rotated at a slow uniform speed, the band meanwhile pressing against the pulley, the reading of the balance A was found to be greatly in excess of that of B. Upon raising the balance B, both readings increased, but that of A was still in excess of that of B. The balance readings indicated the respective tensions in the two straight parts of the flexible band. Denoting the pull given by the balance A as T lb. and that by B as S lb., the results were as follows :—

Force T (lb.)	3·6	5·4	7·5	9·4	11	13·3	16·9
Force S (lb.)	2	3	4	5	6	7	9

Plotting these numbers, a straight-line graph (Fig. 89) resulted of which the equation was found to be

$$\text{force T} = 1{\cdot}87 \times \text{force S}.$$

If, therefore, in the case of an actual driving belt the resisting torque of the driven shaft is increased until slipping of the belt occurs, the final tension in the tight side bears a ratio to that in the

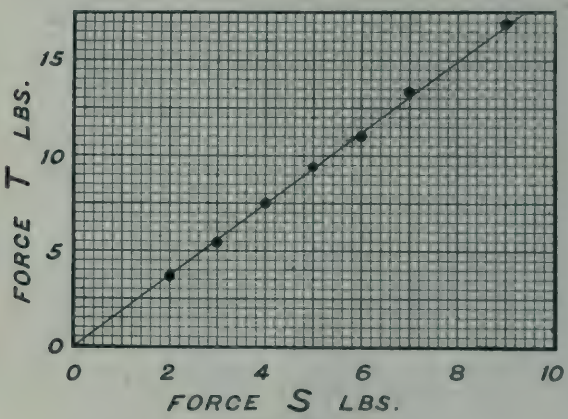

Fig. 89.

slack side which is independent of the initial tension in the belt so long as the arc of contact between the belt and the pulley does not alter. "Initial tension" means the tension common to the two parts of the belt when one at least of the two pulleys offers no resistance to turning.

Without altering the arc of contact, we may vary the diameter of the pulley. Modifying the apparatus of Fig. 88 by substituting a stepped pulley for the one shown, the results obtained were :—

Pulley diameter (in.)	8·15	6·65	5·25	3·8	3·25
Force T (lb.)	8·5	8·5	8·5	8·5	8·5
Force S (lb.)	1·5	1·5	1·5	1·5	1·5
Ratio $\frac{T}{S}$	5·7	5·7	5·7	5·7	5·7

These numbers indicate that the ratio $\frac{T}{S}$ is in no way affected by the diameter of the pulley. It must be remembered, however, that this statement no longer holds good whenever the band offers a material amount of resistance to bending.

After a further modification of the apparatus had been made, the values of the forces T and S were found for different angles of contact. The angle of contact is the angle at the centre of the pulley subtended by the arc of contact between the band and the

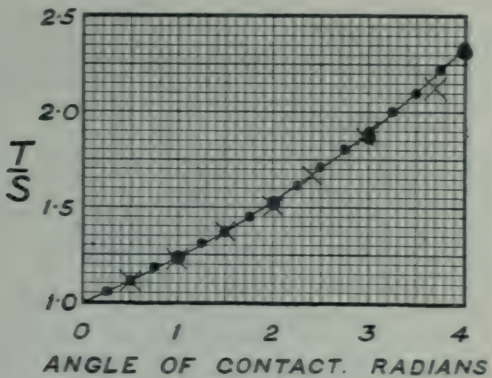

FIG. 90.

pulley. Throughout the experiment the force S was maintained at 5 lb. The results obtained were :—

Angle of contact (radians) .	0·5	0·995	1·5	2	2·4	2·96	3·7
Force T (lb.) . . .	5·55	6·1	6·9	7·5	8·4	9·4	10·6
Ratio $\frac{T}{S}$ . . .	1·11	1·22	1·38	1·5	1·68	1·88	2·12

The points which indicate the values of the ratio $\frac{T}{S}$ plotted against the angle of contact are represented by crosses in Fig. 90. The majority of the points in question are close to the graph, which, although it was drawn independently of the crosses, may reasonably be regarded as passing through the points determined by them. The graph is curved in such a manner as to show that the ratio $\frac{T}{S}$ not only increases as the angle of contact becomes greater, but that it does so at an increasing rate.

In order to obtain data for treating the matter in a different way another experiment was performed upon the pulley and band used

in obtaining the last table of results. For this purpose the arrangement shown in Fig. 91 was employed. A bell-crank lever carries a roller A at the end of its vertical arm. At the end of the horizontal arm, which is equal in length to the vertical one, a load of W lb. is supported. The roller exerts a horizontal force equal to W lb. upon a small slab of wood. The slab transmits this pressure in a direction normal to the face of the pulley through the band which, hanging vertically from a spring-balance, supports a load M.

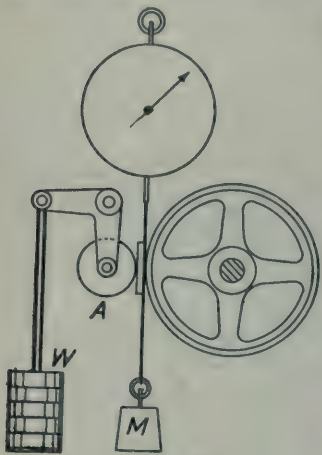

FIG. 91.

When the pulley is turned in an anti-clockwise direction the tangential resistance of friction drags the belt downward. The slab moves down with the belt, turning the roller as it does so. The balance pull becomes greater than the load M, attaining its maximum when the pulley slips over the band.

The difference between the indication of the balance and the load M represents the limiting resistance of friction when the normal pressure between the band and the pulley amounts to W lb. Reversing the direction of turning, the balance pull now becomes less than W, but the limiting resistance of friction is, as before, the difference between the load M and the balance pull. Six different values of W being obtained—averaging in each case the upward and downward indications—a graph was plotted from which the coefficient of friction was found to be 0·21.

Regard the circular arc AB in Fig. 92 as part of the pulley rim; the thickened part represents a very short length of the band wrapped around it. Equilibrium of this short piece takes place under the action of three forces, viz.:—R a pressure exerted by the face of the pulley rim, a force T pulling at one end, and a force T_1 pulling at the other. When the belt is at the point of slipping, the angle θ—between the direction of the force R and the radial line which is normal to the surface of the pulley at the point where R is acting—is the limiting angle of friction. In the case under consideration the tangent of the angle θ is 0·21, for the tangent of the limiting angle of friction is equal to the coefficient of friction.

The pulls T and T_1 are tangential to the surface of the pulley, and hence their directions may be set out. The amount of any one of the three forces being given, a triangle of forces C may be drawn to determine the other two.

Suppose that ϕ is the total angle of contact between the belt and the pulley. With ϕ equal to 4 radians a diagram is drawn on the

left of the figure in which the part of the belt in contact with the pulley is divided into 16 equal arcs. A line to represent the direction of the force R is set out at the centre of each arc. The force T acting on the end of the first portion of the band is the pull along the tight part. Taking T as equal to 10 lb. a triangle of forces C_1 is drawn for the first arc.

Equal tangential forces, acting in opposite directions, produce a tension in the belt at any cross *section* and hence the force which corresponds to T_1 for the first arc is equal to the force corresponding to T for the second arc. The triangle D for the second arc may therefore be built upon the triangle C_1. Proceeding in this manner, a triangle is obtained for each of the 16 arcs, the final line E representing the tension S lb. in the slack part of the belt. Two sides

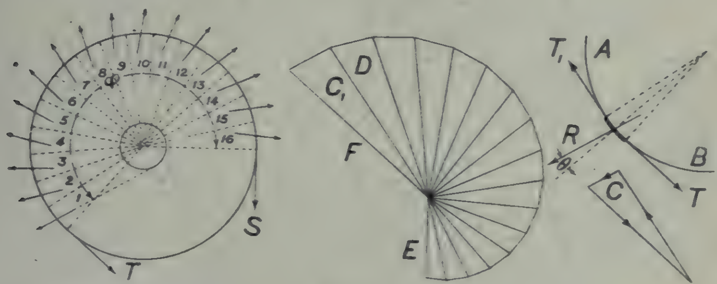

FIG. 92.

of each triangle are parallel to tangents to the pulley rim, but it is easily perceptible that the tangents in question need not actually be drawn.

Suppose the angle ϕ to be gradually diminished by unwrapping the belt and that this is done by eliminating first the 16th arc, then the 15th, and so on. The triangles corresponding to the eliminated arcs would in consequence disappear from the vector figure.

The word "vector" is a convenient term of reference for a line of which the inclination corresponds to the direction of some force, velocity, acceleration, quantity of momentum, etc., an arrow-head indicating whether the quantity in question acts up or down, to the right or the left. The length of such a line is determined by a scale so as to represent a given amount of force, velocity, acceleration or whatever quantity is in question.

After each alteration, S is represented by the last remaining radial vector and the ratio $\frac{T}{S}$ may be found by dividing the length of that vector into the length of the longest one F. In this manner the following table was obtained by measurements from the diagram reproduced in Fig. 92.

Angle of contact (radians)	0·25	0·5	0·75	1·0	1·25	1·5	1·75
Force S (lb.)	9·5	9·0	8·5	8·1	7·65	7·3	6·9
Ratio $\frac{T}{S}$	1·05	1·11	1·176	1·234	1·308	1·37	1·45

Angle of contact (radians)	2·0	2·25	2·5	2·75	3·0	3·25	3·5	3·75	4·0
Force S (lb.)	6·53	6·2	5·85	5·55	5·26	5·0	4·75	4·5	4·26
Ratio $\frac{T}{S}$	1·53	1·61	1·71	1·8	1·9	2·0	2·1	2·22	2·35

The numbers in the last table were employed to plot the graph in Fig. 90 which, it is obvious, agrees with the relationship indicated by the crosses and resulting from the preceding experiment.

The graphical method, however, gives merely an approximation to the truth, for the tension in the curved part of the belt varies *continuously* from point to point. In accordance with this fact it is necessary in order to obtain an exact relationship between the forces T and S to regard the curved portion of the band as divided into an infinite number of parts. If this be done it may be established by mathematical reasoning that

$$\frac{T}{S} = \epsilon^{\mu\theta}$$

when ϵ is a number, 2·302 approximately, which is the base of the hyperbolic system of logarithms, μ is the coefficient of friction between the belt and the pulley and θ is the angle of contact in radians. By taking the common logarithms of each side of the equation we obtain

$$\log_{10} \frac{T}{S} = \mu\theta \log_{10} \epsilon = 0{\cdot}4343\ \mu\theta.$$

Since the expression on the right-hand is the common logarithm of $\frac{T}{S}$ it is only necessary to obtain the antilogarithm in order to find $\frac{T}{S}$. The following table represents values of the ratio for various angles of contact which were calculated in this manner.

Angle of contact (radians)	1	2	3	4
$\log_{10} \frac{T}{S}$	0·0912	0·1824	0·2736	0·3648
Ratio $\frac{T}{S}$	1·234	1·522	1·878	2·316

The numbers representing $\frac{T}{S}$ are plotted in Fig. 90, giving points which are indicated by the larger circles. The graph shows that the experimental, the graphical, and the mathematical methods concur —within certain limits of error—in giving the same result, but the fact should not be overlooked that the experiment was conducted at an extremely low speed of rubbing.

It must be understood that Fig. 90 shows merely the *limiting* value of $\frac{T}{S}$ for all arcs of contact within the range of the horizontal scale. In actual examples of belt-driving $\frac{T}{S}$ may have any value which is less than the graph gives, but so long as μ does not exceed 0·21, $\frac{T}{S}$ cannot possibly be greater; for a resisting torque which would occasion it to be so is sufficient to cause the belt to slip past the pulley.

Knowing the limiting value of $\frac{T}{S}$ and the greatest working load the belt will sustain, the greatest possible effective tension may be found by taking T equal to the greatest working load. Knowing the greatest effective tension possible, the H.P. which a belt will transmit at any given speed is readily obtained.

A belt passing around a pulley has an angle of contact of 200° and the coefficient of friction is 0·28. Find the value of the fraction $\frac{T}{S}$ when T = tension in the tight side and S = tension in the slack side of the belt.

$$\text{Log. } \frac{T}{S} = 0{\cdot}4343 \times 0{\cdot}28 \times \frac{200}{57{\cdot}3} = 0{\cdot}424.$$

$$\text{Antilog. of } 0{\cdot}424 = \frac{T}{S} = 2{\cdot}65.$$

What are the numerical values of T and S when $T - S = 600$ lb. and $\frac{T}{S} = 3$?

Because $$\frac{T}{S} = 3,\ S = \frac{T}{3}$$

$$\therefore T - S = T - \frac{T}{3} = \frac{2}{3}T$$

$$\therefore \frac{2}{3}T = 600 \text{ lb.}$$

$$T = \frac{1{,}800}{2} \text{ lb.} = 900 \text{ lb}$$

$$S = \frac{900}{3} \text{ lb.} = 300 \text{ lb.}$$

A belt-pulley is required to transmit 6 H.P. at 150 revs. per min. The coefficient of friction between the belt and the pulley is 0·28. The belt embraces an arc of the pulley-rim which subtends an angle of 3 radians. If the greatest allowable pull in the belt is 160 lb., what minimum diameter of pulley is necessary?

$$\text{Log. } \frac{T}{S} = 0{\cdot}4343 \times 0{\cdot}28 \times 3 = 0{\cdot}364.$$

$$\frac{T}{S} = \text{antilog. of } 0{\cdot}364 = 2{\cdot}312.$$

$$T - S = 160 - \frac{160}{2{\cdot}312} = 90 \text{ lb.}$$

If R ft. be the radius of the pulley

$$\frac{(T - S)\, R \times 2\pi 150}{33{,}000} = 6.$$

$$\therefore R = \frac{33{,}000 \times 6}{90 \times 300\pi} = 2{\cdot}33 \text{ ft.}$$

$\therefore$ Diameter of pulley $= 2R = 4{\cdot}66$ ft.

EXAMPLES.

86. An engine with its tender weighs 80 tons; when moving along a level track at the rate of 20 miles an hr. against a resistance of 7 lb. per ton, at what H.P. is it working?
87. One hundred tons of water per hr. pass over a water-fall 72 ft. high. If all this water were diverted through a turbine which had an efficiency of 88 per cent., what would be the effective H.P. of the turbine?
88. A jet of water issuing from the nozzle of a fire-engine hose has a sectional area of 1 sq. in. and a speed of 150 ft. per sec. To what effective H.P. is the jet equivalent?
89. The piston of a steam-engine has a diameter of 8 in. The piston-rod has a diameter of 1·25 in. The mean effective pressure on each side of the piston is 40 lb. per sq. in. The crank is 6 in. long. When the crank shaft rotates at 200 revs. per min., what is the I.H.P. of the engine?
90. A simple steam-engine, double-acting, that is, having two working strokes to each revolution, has a fly-wheel 6 ft. in diameter. The driving belt of the engine passes round the fly-wheel and when the engine is at work the effective tension in the belt is 240 lb. The crank of the engine is 6 in. long.

Find the average total effective pressure of the steam upon the piston.

91. A steam-engine develops 20 I.H.P. The crank-pin circle has a diameter of 15 in. and the engine makes 220 revs. per min. What force is exerted upon the piston?

92. The average crank effort exerted on the crank-pin of a double-acting engine is 1,000 lb. The length of the crank is 8 in. The crank-shaft rotates at the rate of 120 revs. per min. Find the H.P. of the engine.

93. The mechanical efficiency of a steam-pump is 70 per cent. The pump delivers 120,000 gall. of water per hr., raising the water through a height of 40 ft. Find the I.H.P. of the pump.

94. In testing a steam-pump a screw-down valve was placed in the delivery pipe, at a height of 35 ft. above the level of the water in the suction well. A pressure-gauge, connected to the pipe between the pump and the valve, was fixed at the same height as the latter. The valve being partially open, water was forced past it at the rate of 15,000 gall. in 20 min., the gauge meanwhile indicating a pressure of 62 lb. per sq. in. Determine the W.H.P. of the pump.

95. A boiler feed-pump, fixed at the same level as the boiler, is used to take water from an elevated tank and force it into the boiler. When the water-level of the tank is 36 ft. above the level of the water in the boiler, the steam pressure is 60 lb. per sq. in., and water is pumped in at the rate of 9,600 lb. per hour; what is the W.H.P. of the pump?

96. A pump has an efficiency of 60 per cent. and raises 1,000 gall. of water through a height of 26 ft. in 10 min. Calculate the H.P. developed by the pump.

97. A steam-pump has a piston 5 in. in diameter. The length of the stroke is 1 ft. and 120 strokes per min. are made. The height of the lift is 85 ft. and the quantity of water delivered per min. is 30 cub. ft. The steam-pressure is 95 lb. per sq. in. Find the mechanical efficiency of the pump.

98. A single-acting steam-pump has a plunger of 3 in. diameter which makes 90 strokes of 8 in. length per min. If the water delivered is forced into a boiler working at a steam pressure of 120 lb. per sq. in., what is the effective H.P. of the pump?

99. The driving pulley C of a Tatham dynamometer (Fig. 86) has a diameter of 2 ft. and the driven pulley D has a diameter of 15 in. The latter pulley has a speed of 120 revs. per min. When the apparatus is in use, a spring-balance G, of which the line of pull is 18 in. from the knife-edge of the lever, indicates a pull of 60 lb. What H.P. is transmitted?

100. In Fig. 86 assume the following: The balance G indicates a pull of 34 lb. and its axis is 2 ft. from the knife-edge at F; the centres of the guide pulleys EE are 18 in. apart. Each

guide pulley has a diameter of 6 in. The pulley D rotates at 300 revs. per min. What H.P. is transmitted?

101. In the arrangement shown in Fig. 86 find the tension in the slack part of the belt from the following data: Tension in tight part 84 lb.; pull along balance, 42 lb.; distance between guide-pulley centres, 16 in.; distance between knife-edge and axis of balance, 22 in.

102. Suppose that a dynamometer as shown in Fig. 86 has a lever arm 2 ft. long which is balanced by a pull at the end of 80 lb. when the appliance is at work. Assume that the pulley immediately below the knife-edge has a diameter of 12 in. The pulley below this has a diameter of 20 in. The larger pulley rotates at 120 revs. per min. What H.P. is passing through the apparatus? (It is assumed that the belt does not slip.)

103. In Fig. 82, C, D, and E are spur-wheels. C has a diameter of 10 in. and is mounted at one end of a bell-crank lever which turns freely about the shaft on which the wheel E is fixed. Power is applied to the shaft upon which the wheel D is keyed. The bell-crank lever is prevented from rotating by a pull of 50 lb., at the end of the horizontal arm of the lever, which is 30 in. long. The wheel E is 20 in. in diameter and revolves at a speed of 100 revs. per min. What H.P. is transmitted?

104. Two shafts are connected by a belt passing over pulleys each 2 ft. in diameter. At a speed of 100 revs. per min, when the greatest tension that the belt will sustain is exerted along it, slipping is on the point of taking place. The H.P. then transmitted amounts to 8·6. If the pulleys are changed for a pair, each of which is 3 ft. 2 in. in diameter, what is the greatest H.P. that may then be transmitted? The other conditions remain unaltered.

105. Refer to Fig. 88. Assume that the balance A indicates a pull of 54 lb. If the coefficient of friction between the band passing over the wheel and the surface of the wheel is 0·32, what pull will be indicated by the spring-balance B?

106. In an absorption dynamometer, a cord passes over a wheel, each end being attached to a spring-balance. The coefficient of friction between the cord and the wheel is 0·35, and the balances indicate pulls of 7·5 lb. and 76 lb. respectively. Find the angle of contact between the cord and the wheel.

107. A belt passing over a pulley has an angle of contact of 190°, and the coefficient of friction between the pulley and the belt is 0·38. If the ratio of the pull on the tight side to that on the slack side is at its highest value, what will these pulls be when the effective tension is 240 lb.?

108. When the balance A in the arrangement shown in Fig. 88 indicated 11 lb., the balance B indicated 6 lb. What was the

coefficient of friction between the band and the rotating pulley ?

109. A belt pulley is to transmit 8 H.P. at a speed of 200 revs. per min. The arc of contact between the pulley and the belt subtends an angle of 118° at the centre of the pulley. The greatest allowable pull in the belt is 200 lb. If the coefficient of friction between the pulley and the belt is 0·35 what is the minimum diameter of the pulley ?

CHAPTER VI

Screw-Cutting.—A diagrammatic representation of a screw-cutting lathe is given in Fig. 93. A belt-driven cone-pulley causes a spindle or mandrel to rotate and carry around with it a cylindrical piece of metal A. Keyed to the mandrel, and therefore turning at the same rate as the metal A, is a toothed wheel B termed the

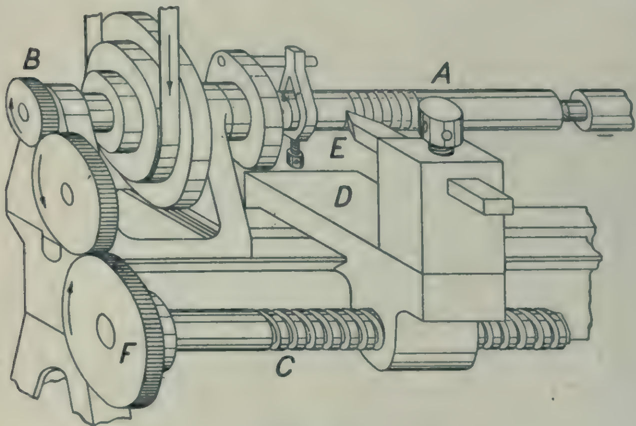

FIG. 93.

mandrel wheel. A long *leading screw* or *guide screw* C works in a nut attached to the saddle D; when the screw is turned round a cutting tool E will therefore move along the lathe bed.

The spur wheel F, fixed to the leading screw, is connected to the mandrel wheel either by a single idle wheel, as shown, by two idle wheels, or by a compound wheel-train. Any wheel may be replaced by another with a different number of teeth; hence the term *change wheels.* In a *simple train,* such as the one in the figure, each wheel turns about an independent axis. When the cylinder A and the leading screw have, as shown, the same direction of rotation, the screw produced on A is of the same "hand"—right or left—as the leading screw. An additional idle wheel, or the absence of any idle wheel, will result in different directions of rotation and consequently in different "hands" of the screws. A screw which requires, say, eight turns in order to move a nut 1 in. is said in workshop language to possess "eight threads per inch." The actual number of *screw threads* may be one or more. The above expression must therefore

be regarded as meaning "*eight turns per thread per inch.*" It is convenient, however, to employ the former phrase.

If the leading screw possesses, say, "four threads per inch," it must be turned round four times to traverse the tool 1 in. If a screw possessing twelve threads per in. is required, the cylinder A must turn round twelve times whilst the tool travels 1 in. Consequently the rotational speed of A bears a ratio to the rotational speed of C of 12 : 4. The rotational speeds are in inverse ratio to the numbers of teeth on the wheels turning respectively with A and C. Hence, when a simple train of wheels is used

$$\frac{\text{number of threads per in. to be cut}}{\text{number of threads per in. on leading screw}} = \frac{\text{number of teeth on leading screw wheel}}{\text{number of teeth on mandrel wheel}}.$$

If a simple wheel-train were arranged to cut a large number of threads per in., the leading screw would generally require a wheel with an inconveniently large number of teeth. This is avoided by using a compound wheel-train as shown in Fig. 94, in which M is the mandrel wheel, L is the leading-screw wheel, and A and B are intermediate wheels locked together and turning upon the same pin.

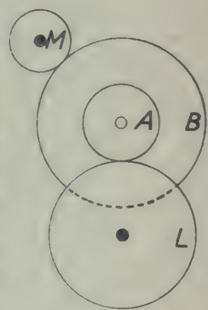

FIG. 94.

For a compound train the right-hand side of the last equation must be replaced by the expression,

$$\frac{\text{product of tooth numbers of followers}}{\text{product of tooth numbers of drivers}},$$

the mandrel wheel being the first driver and the leading-screw wheel being the last follower. A typical set of change wheels consists of 22 wheels, the smallest possessing 20 teeth and the largest 120 teeth. Two of the wheels in the set possess 60 teeth. Excepting the extra wheel, the tooth numbers of the wheels progress by 5 at a time from the smallest to the largest. For a compound train, any wheels may be chosen which can be so arranged as to satisfy the equation :—

$$\frac{\text{number of threads to be cut}}{\text{number of threads on leading screw}} = \frac{\text{product of tooth numbers of followers}}{\text{product of tooth numbers of drivers}}.$$

The leading screw of a lathe has 3 *threads per in. The change wheels available have* 20, 30, 40, 55, *and* 100 *teeth respectively. A train is to be arranged to enable a screw of* 11 *threads per in. to be cut.*

From the last equation we see that tooth numbers must be chosen to form a fraction of the value $\frac{11}{3}$; four wheels may be employed as shown in Fig. 94. As the denominator has the lesser value, the 20 and 30 tooth wheels may be chosen provisionally for that part of the fraction. This will necessitate a numerator of 200×11 or 2,200, because 20×30 is 200 times the denominator of the fraction $\frac{11}{3}$. Of the remaining wheels those possessing 55 and 40 teeth respectively will satisfy the requirement. Hence,

$$\frac{\text{product of tooth numbers of followers}}{\text{product of tooth numbers of drivers}} = \frac{55 \times 40}{20 \times 30} = \frac{11}{3}.$$

Referring to Fig. 94 we may arrange the train thus: M 20 teeth, B 55 teeth, A 30 teeth, and L 40 teeth.

Linkages.—The connecting rod of a steam-engine possesses freedom of turning at each end; in this it differs from a crank which has one end rigidly attached to another piece of metal. In consequence, any force transmitted through the connecting rod must be a direct pull or push, whilst the crank transmits force chiefly by virtue of its capacity to resist bending. The crank is an example of a lever-arm whilst the connecting rod is an example of a *link*.

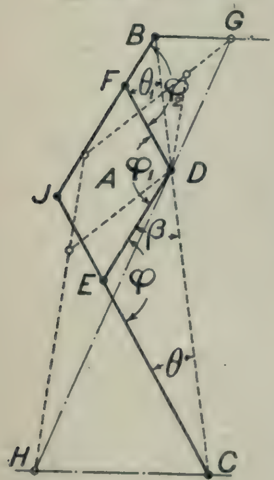

FIG. 95.

If three links are connected together by pin-joints, a rigid frame results. As an element of a machine such a frame can be regarded only as a single part. If five links are joined up, one of them may be held at rest whilst one of the adjoining links is moved. Of the remaining three links one may or may not remain at rest. One thing is certain; none of the remaining three is compelled to move in a manner definitely related to the movement of the second link.

Join together four links, and then if one be held at rest, the movement of any one of the others always results in the movement of the two remaining; moreover, the movement is such that the path of any point in any one of the links is determinate; that is, only one particular path is possible for each point. A motion of this kind, where each point in a machine part is able only to pursue a particular path, is said to be *constrained*. A combination of

parts, in each of which only constrained motion is possible, is termed a *mechanism*. By extending somewhat the meaning of the word "link" we may regard any machine as an assemblage of links.

In Fig. 95 the parallelogram A is formed by joining four links. The link JF is extended to the point B, and the link JE to the point C so that the points BDC lie in a straight line. Fixing the point D and moving the end C in the line HC, the end B will move in the line BG. The lengths BF, FD, DE, and EC are unalterable. The triangles BFD and DEC are similar. When movement of the end C takes place the angles θ, ϕ, and β will alter in size, but because

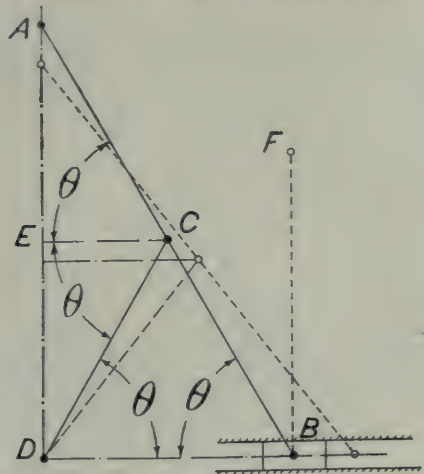

FIG. 96.

FD and JE and also FJ and DE are always parallel, the angles ϕ, ϕ_1, and ϕ_2 will always be of equal size. Since also

$$BF : FD :: DE : EC$$

any pair of triangles corresponding to BFD and DEC which result from the movement of the point C will be similar. Hence the angles θ and θ_1 will always be equal. Since the sum of the angles θ, ϕ, and β will always equal two right-angles, then the sum of θ_1, ϕ_1, and β will also equal two right-angles in every case. Consequently the points BDC will always be in a straight line. D being a fixed point, any such line as GH corresponding to BC is divided at D into parts which have the same ratio as BD and DC. Hence any two triangles corresponding to BGD and CHD will be similar; therefore

$$BD : DC :: BG : HC$$

and the line BG which is the path of the point B must be parallel

to HC. The speed of the point B will always bear the same ratio to the speed of the point C as BD : DC.

This arrangement of links is termed a *pantagraph;* it is often employed to form a "reducing gear" for attaching an indicator to an engine (see p. 115); in which case the end C is attached to the crosshead and the end B to the cord which operates the indicator. The arrangement is sometimes embodied in the indicator itself in order to cause a pencil to move in a straight line.

In Fig. 96 AB is a rod jointed at the centre to a link CD having a length equal to half that of AB. The end D turns about a fixed pin. The end B turns about a pin fixed to a block free to slide along a horizontal groove. Any movement of the block along the groove results in the end A moving along a vertical straight line passing through D.

The angles marked θ are all of equal size. A movement of the

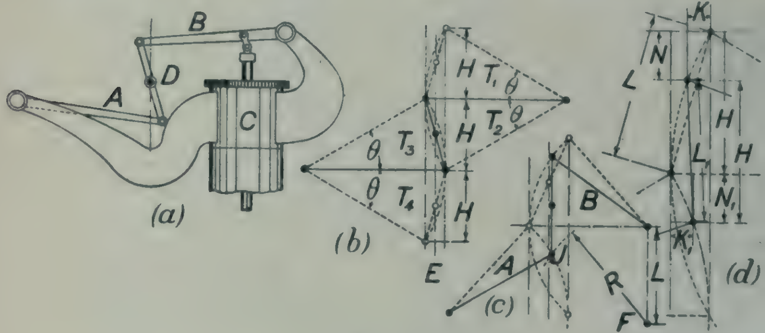

FIG. 97.

combination results in altering the size of these angles, but they will still be equal. The triangles ACE and DCE, which are similar and equal, will be replaced by another pair of equal and similar right-angle triangles which in every case will have in common the line corresponding to EC. As E always denotes a point in a vertical line through D, the end A of the rod must also be in the same line for any position of the rod. This mechanism is known as the *Scott Russell parallel motion;* it was designed to provide a straight-line movement for the end of a piston-rod which was attached at the point A. The sliding block may be replaced by a link FB, pivoted at F. Since B will in this case move in the arc of a circle the motion of the point A will be only approximately straight.

In Fig. 97 (*a*), two light arms A and B are pivoted at the ends of the two curved brackets attached to the swivelling sleeve C of a steam-engine indicator. The free ends of the arms are joined by a link D. At the centre of the link a pencil is fixed which is required to move upward along a vertical straight line when the piston of the instrument rises. (A complete section through an indicator provided

with a different linkage for transmitting motion to the pencil is given on p. 116). Let each of the radial arms A and B be placed horizontally as shown in the diagram (*b*). The distance between the two arms is then H. Let each arm swing through an angle θ above and below its horizontal position. The intersecting circular arcs represent the respective paths of the ends of the arms. The chord of one arc is a tangent to the other arc. The triangles T_1 T_2 T_3 T_4 are similar and equal. The inclination of the link at either extreme position will then be the same as at its mean position. With these data it is not difficult to see that the centre of the link in each of the three positions given lies on a vertical straight line E.

Let L represent the length of the link and R the length of the radial arm. Find a point F (diagram *c*) at a depth L immediately below the pivot point of the arm B. With F as centre, strike an arc of radius R to intersect the path of the end of the arm A. When the link is vertical, J is the position of its lower end. In order to show clearly that the point J and consequently the centre point of the link also are not situated in the same vertical line as the one containing the extreme and mid-position points of the link centre, the lengths of the arms A and B with respect to that of the link are considerably reduced in diagram (*c*). From this diagram it is evident that the path of the link centre is not strictly a straight line, but the deviation from accuracy between the limiting positions indicated in the diagram (*b*) is only slight when the length of the link is a comparatively small fraction of the length of the arm A or B. It is easy to see how the deviation arises. Assume that the end of the arm B moves from its upper extreme position along its circular path until its lateral displacement is K (diagram *d*), and its vertical displacement is N. If the link centre is to move accurately along the line E (diagram *b*) an equal lateral displacement K_1 in an opposite direction must be given to the lower end of the link ; if this end moves along the curved path of the end of the arm A, the resulting vertical displacement N_1 must equal the displacement N to fulfil the stated condition. Consequently the vertical distance H between the ends of the link will remain unaltered, although the inclination of the link has increased. These conditions are not realised because we cannot have right-angle triangles with perpendiculars and hypotenuses respectively equal but with bases of different lengths. Consequently the length L_1 is less than L, the actual length of the link ; the latter length cannot change and therefore the lower end of the link must occupy a position a little further down, and rather more to the right, than the conditions of accuracy demand. One half of the extra lateral movement of the lower end of the link is the amount of deviation of the link centre. It is obviously possible to obtain a greater vertical movement of the centre point of the link than is indicated by the limiting positions of diagram (*b*) without exceeding the maximum deviation within those limits. This

linkage was devised by James Watt for the purpose of guiding the end of a piston-rod in a straight line, and is known as *Watt's parallel motion.*

Transmission of Rotary Motion.—In Fig. 98 two shafts A and B are shown with their axes parallel and separated by a small distance H. Motion may be transmitted from one to the other by four spur

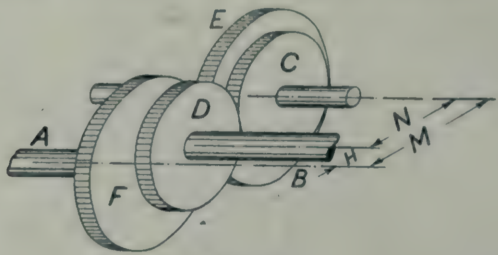

FIG. 98.

wheels arranged as shown. If the shafts are to possess equal rotational speeds, C and D are equal in size and N is equal to the pitch circle diameter of each. The pitch circle diameters of E and F are each equal to M. The wheels C and E are keyed to a counter-shaft parallel to the other two shafts.

Another device for achieving the same result under the same circumstances is shown in Fig. 99, and is known as *Oldham's*

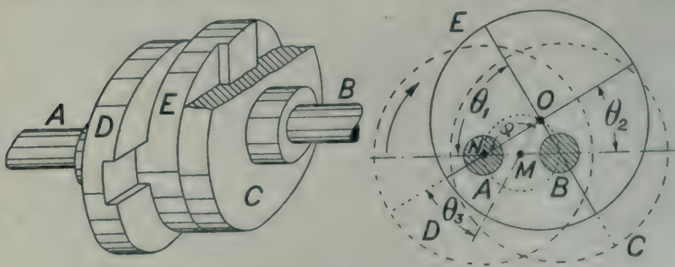

FIG. 99.

coupling. A face-plate C, with a diametral groove, is keyed to the shaft B. A similar face-plate D is keyed to the shaft A. A loose circular plate E is provided with two projecting feathers, one of which slides in each groove. As the feathers are set at right-angles the grooves also must always cross each other at that angle. O is the crossing point of the centre lines of the grooves.

Because these centre lines are always at right-angles and must always pass through fixed points situated in the shaft axes, the path of the point O is a circle. It is evident that the sum of the angles θ_1 and θ_2 is always equal to a right-angle, and that, as the shafts

turn, any increase of one angle is accompanied by an equal diminution of the other. The feathers therefore, and consequently the shafts also, rotate at equal speeds and in the same direction.

Suppose the centre O of the plate E to be connected by a link to the centre M of its circular path. For any position of the plate E, the triangle OMN is isosceles and hence the angle θ_3 decreases at the same rate as θ_2. The increase in the angle ϕ being equal to the aggregate decrease of θ_2 and θ_3, ϕ increases at a rate which is twice the rate of decrease of θ_2, or increase of θ_1. The rate of rotation of the plate E about the centre M is the same as that of the imaginary link and is therefore double the rate of rotation of the shafts about their respective axes. In order more vividly to realise this motion, assume that the groove on the left-hand shaft is vertical, the centre of the plate E then coinciding with the axis of the left-hand shaft. Let the groove now turn through a right-

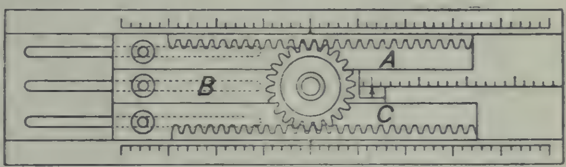

FIG. 100.

angle ; it is easily seen that during this movement the centre of the plate turns through a semi-circle (or two right-angles) in moving to the centre of the other shaft. In addition to its rotation about the point M the plate turns about its moving centre O at the same rate as either shaft about its own axis ; for any change in the angle θ_3 indicates turning of the feathers about the outer end of the imaginary link.

Rolling Motion.—Fig. 100 represents a piece of apparatus in which three strips A, B, and C are arranged to slide along a frame to which each may be clamped. A pinion may turn about a pin fixed on B or be prevented from turning by tightening a milled nut. The pinion engages with two racks, one fixed to the strip A, the other to the strip C. Each strip is provided with an index finger. When a strip is moved along the frame its index finger moves over a scale of length.

Set each finger to the zero of its scale. With all the strips free but with the pinion fixed, move the strip B. It is obvious that the strips A, B, and C have a common speed. Hence,

$$\text{speed of A} + \text{speed of C} = 2 \times \text{speed of B}.$$

Fixing the lowest strip and allowing the pinion freedom to turn, impart a movement of 1 in. to the strip B. The resulting movement

of A is 2 in. When B is moved 2 in., A moves 4 in. Whatever movement is made by B, that of A is twice as much. Since the movement of each strip is made in the same time, and since C has no movement, again

$$\text{speed of A} + \text{speed of C} = 2 \times \text{speed of B}.$$

Now let all three strips and the pinion as well be free to move. Push B and C to the right simultaneously, but move B rather more quickly than C. C being moved 1 in. in this way whilst B is moved 2 in., A it is seen moves 3 in. The speeds as before are proportional to the movements that occur, and the equation which has already been given twice is still applicable.

Relative Motion.—If all three points A, B, and C in a wheel of a vehicle (Fig. 101) were moving horizontally at the same speed V, say, sliding instead of rolling upon the ground would take place. For the instant that the point C in a rolling wheel touches the ground, it is at rest; if the centre B moves forward at the speed V the highest point A has a horizontal velocity V_1 which is equal to 2V because the vertical diameter AC is equivalent to a link turning about the point C. The relationship between these velocities is based upon the assumption that the point C rests upon a stationary surface and that each velocity is a rate of displacement with respect to it.

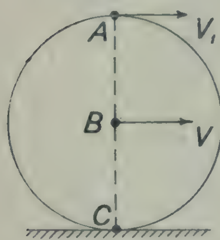

FIG. 101.

A velocity is generally assumed to be with respect to the earth's surface, any motion which the earth may possess being ignored. A fly, or some other insect, situated at the centre B of the rolling wheel (Fig. 101), might with equal justice disregard the motion V of the wheel. From the insect's standpoint, the point A would have a velocity V, whilst both the point C in the wheel and the ground itself would be moving with a velocity V to the left, that is, — V.

If an arbitrary standpoint such as B in Fig. 101 is selected and the velocities of other points stated with regard to it, these velocities are said to be *relative to the point B*. The point C, which is momentarily at rest on the ground, has therefore a velocity — V relatively to the point B, whilst the point B has a velocity V relatively to the point C. The relative velocity of the point A with respect to B is also V. Velocities stated with respect to the earth's surface are said to be *absolute*. Hence the absolute velocity of A is 2V, of B it is V, and of C it is nothing.

Rolling Motions in Mechanism.—The most usual application of rolling motion in mechanism consists in causing one wheel to roll round another. In Fig. 102 A, B, and C indicate three bevel pinions "in mesh." A turns freely about a gudgeon pin which is con-

tinued outward to form a winch D. The pin is fixed rigidly in a piece of metal F which turns around a spindle E. To this spindle

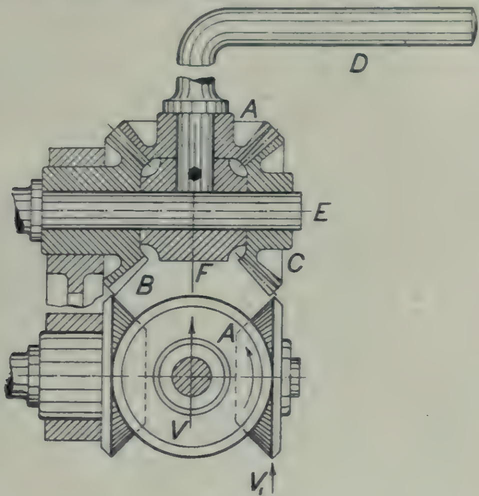

FIG. 102.

the pinion C is keyed, whilst the remaining one, B, is prevented from turning by being fixed to a frame.

When the winch is turned, the pinion A rolls around B and consequently the point in the pitch circle of the former which is momentarily on the extreme right-hand has a tangential velocity V_1 amounting to 2V, when V is the speed of the centre of the pitch circle. The velocity 2V is communicated to the pitch circle of C and consequently the spindle E rotates at twice the rate of the winch. Compare the lower part of the figure with the apparatus of Fig. 100. The arrangement is used for imparting motion to the driving wheel of a fan, a cream separator, or any other machine where a high rotational speed is necessary.

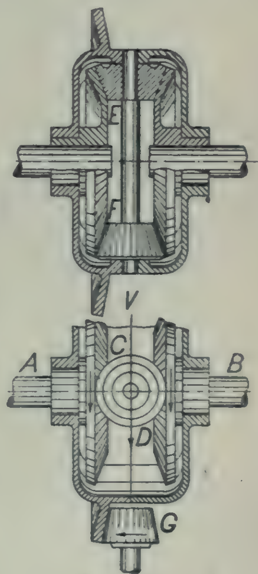

FIG. 103.

Another example of rolling motion occurs in the *differential gear* which is fitted to the driving axle of a self-propelled vehicle. Fig. 103 represents such a gear in plan and elevation. The driving axle AB is in two parts. Bevel wheels C and D are keyed one to each part.

Two or more bevel pinions, as E and F, gear with the wheels C and D. Each of a pair of these pinions runs loosely on a spindle which is fixed inside a circular box free to turn about the axle AB. The box is rotated by a pinion G which engages with a ring of teeth formed on the box. The spindle carrying the pinions E and F

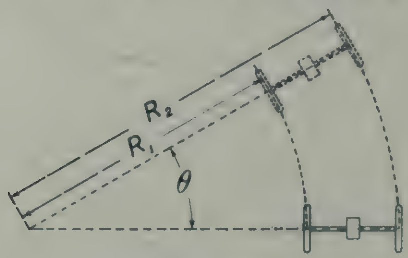

FIG. 104.

rotates in a vertical plane. The movement imparted to the wheels C and D depends upon circumstances.

Suppose the paths of the road wheels of the vehicle to be shown by the dotted circular arcs in Fig. 104. When the axle moves

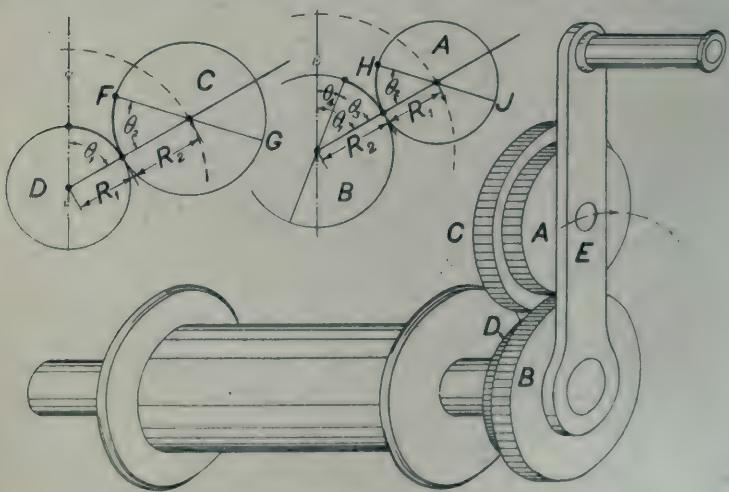

FIG. 105.

through the angle θ, one of these wheels describes a path of which the length is θR_1; the length of the path described by the other is θR_2. These lengths are obviously unequal and hence if no sliding occurs between either wheel and the ground, the part of the axle to which one driving wheel is fixed will not rotate about its axis through the same angle as the other part. The object of the differential gear is to allow equal torques to be transmitted to the

two parts of the axle whilst permitting a difference of rotational speed between them.

Compare the lower part of Fig. 103 with Fig. 100 and it will not be difficult to see that for the instant in which one of the spindle pinions occupies any particular plane, that pinion is related to the two bevel wheels C and D in exactly the same way that the pinion in Fig. 100 is related to the two racks. The linear speed V of the centre of the pitch circle of the pinion E (Fig. 103)—relatively to the shaft axis—is determined by the driving pinion G. When the vehicle pursues a straight path the tangential speeds of the pitch circles of C and D—relatively to the shaft axis—are each equal to V. The pinions E and F do not then rotate about the spindle. Let the vehicle, however, turn a curve; the tangential speeds of C and D consequently become unequal and the pinions E and F will rotate about the spindle. If the tangential speeds are V_1 and V_2 respectively,

$$V_1 + V_2 = 2V,$$

V being the linear speed of the centre of the pitch circle of the spindle pinion.

A clear conception of this matter will be formed only when the fact is grasped that any difference of rotational speed between C and D is determined by a difference of rotational speed between the two road wheels, not *vice versa*.

A peculiar arrangement of toothed wheels for operating a winding barrel is given in Fig. 105. The wheels A and D are of equal size. The wheels B and C are also of equal size, each being rather larger than A or D. A and C are rigidly connected, and turn upon a stud fixed to the radial arm E. The wheel D has no movement whatever; B is keyed to the barrel spindle and the arm E turns freely about the barrel spindle.

Let the arm E be moved through an angle θ_1. The result is that the pitch circle of C rolls upon that of D and the diameter FG of the wheel C, which was initially vertical, assumes the position shown in the figure. R_1 and R_2 being the respective radii of the pitch circles,

$$\theta_1 R_1 = \theta_2 R_2,$$

for each of the expressions above represents the length of either of the thickened arcs of rolling. Hence,

$$\theta_2 = \frac{R_1}{R_2}\theta_1.$$

Because A and C are fixed together the diameter HJ of the wheel A has also moved through the angle θ_2 with respect to the arm E. The length of the arc of rolling of the wheel A is therefore $\theta_2 R_1$. Had the wheel A been fixed rigidly to the arm E and the wheel D

been free to turn, the rotation of B would have been through the angle θ_1, but the rotation about its own centre of the wheel A has caused a backward rotation through an angle θ_3 to be imparted to the wheel B. The arcs of rolling between A and B being equal,

$$\theta_3 R_2 = \theta_2 R_1,$$

from which

$$\theta_3 = \frac{R_1}{R_2}\theta_2 = \frac{R_1^2}{R_2^2}\theta_1.$$

The net rotation of the wheel B, and therefore of the barrel also, is through the angle θ_4, and as

$$\theta_4 = \theta_1 - \theta_3,$$

$$\theta_4 = \theta_1 - \frac{R_1^2}{R_2^2}\theta_1 = \theta_1\left[1 - \left(\frac{R_1}{R_2}\right)^2\right]$$

The rotational speed of the arm being proportional to θ_1, that of the barrel is proportional to θ_4 and hence,

$$\frac{\text{rotational speed of barrel}}{\text{rotational speed of arm E}} = \frac{\theta_4}{\theta_1} = 1 - \left(\frac{R_1}{R_2}\right)^2$$

Since the numbers of teeth possessed by different sized wheels are in the same ratio as their respective pitch circle radii, it is evident that the value of the last expression will not be altered by substituting tooth numbers for pitch circle radii. The complete device shown in Fig. 105 is known as *Harrison's winch*, but a wheel train in which some of the wheels are attached to a swinging arm is termed an *epicyclic train*.

Referring to Fig. 105, let each of the wheels A and D possess 11 teeth and each of the wheels B and C possess 12 teeth. How many revolutions must the handle make in order to turn the winding barrel round once?

$$\frac{\text{Rotational speed of arm}}{\text{Rotational speed of barrel}} = \frac{1}{1 - \left(\frac{11}{12}\right)^2} = 6{\cdot}25 \text{ (approx.)}.$$

The handle therefore turns 6·25 times.

Variable Velocity Ratio.—In Fig. 106 two views of a friction clutch are given. Two shafts A and B have their axes in a straight line. To A, a circular casting C is keyed. To B, a casting D is keyed. Two curved blocks E and E_1 move in guides which form part of the casting D. A circular sleeve G is capable of sliding along the boss of the part D, movement being caused by a lever

which is not shown. When the sleeve G moves to the left, two links H and H_1, which together form what is termed a *toggle*, cause the blocks E and E_1 to move outward from the axis of the shaft. The object of the arrangement is, by means of a moderate force exerted along the sliding sleeve, to cause a very considerable pressure P to be exerted between the casting C and the blocks E and E_1. When this pressure exists a torque applied to one shaft may be transmitted to the other by virtue of the tangential resistance of friction, energy being transmitted from shaft to shaft when rotation occurs.

The vector triangle J represents the forces acting upon the lower pin of the link H. The triangle K represents the forces which act

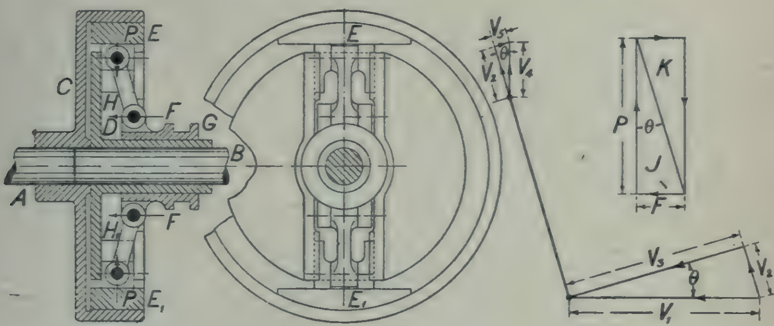

FIG. 106.

upon the upper pin. Let V_1 be the velocity of the lower pin of H when the sleeve is moved to the left. V_1 may be resolved into component velocities, one of them V_3, tending to turn the link around the upper pin, the other V_2 moving the link in the direction of its own centre line. V_2 is a velocity actually imparted to the upper pin, which, having a resultant vertical velocity V_4, possesses also the component velocity V_5 in a direction at right-angles to that of V_2. As tightening of the blocks E and E_1 takes place, the forces exerted act through minute distances. Some energy is then transmitted through the links.

The force F is exerted upon a point which has a velocity V_1 whilst the resistance P acts upon a point which has a velocity V_4. Measuring the vectors in the diagram, we get

$$\frac{V_1}{V_4} = 2{\cdot}98 \quad \text{and} \quad \frac{P}{F} = 2{\cdot}98.$$

This result gives

$$P \times V_4 = F \times V_1.$$

The truth of this equation is readily proved. Comparing the velocity vectors of the lower triangle with those of the upper one, we have

$$\frac{V_1}{V_4} = \frac{V_2}{V_5}.$$

Comparing the force vectors P and F with the velocity vectors of the upper triangle, we have

$$\frac{P}{F} = \frac{V_2}{V_5}.$$

Hence,

$$\frac{P}{F} = \frac{V_1}{V_4} \text{ and } PV_4 = FV_1.$$

FV_1 is the rate at which energy is exerted upon the lower pin, and PV_4 is the rate at which it is transmitted to the curved block E.

$\frac{P}{F}$ is the mechanical advantage and $\frac{V_1}{V_4}$ is the velocity ratio of the combination of which E, H, G and the two pins are the moving parts.

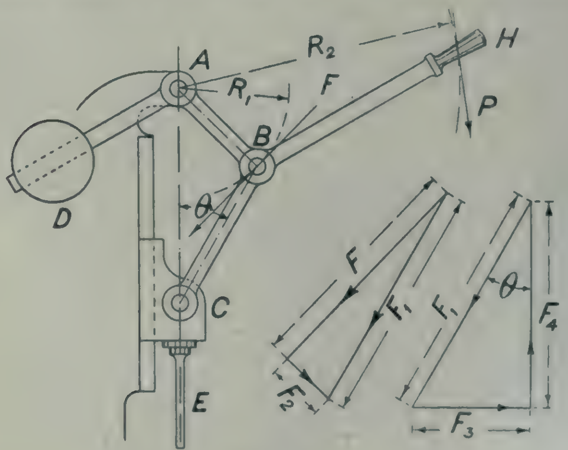

FIG. 107.

The equality of the above two ratios depends upon the assumption that energy may be transmitted along the link without frictional loss. The present instance provides an example of a variable velocity ratio, for the ratio $\frac{V_1}{V_4}$ changes with the inclination of the link.

Fig. 107 shows a mortising machine in which the chisel E is actuated by a toggle. A bent lever DABH turns about a pin at A. A link BC connects the lever to a sliding block C which carries

the chisel E. A balance weight D has the effect of maintaining the block C in its upper position until the handle H is forced down by the application of a force P.

When this happens, the resistance of the chisel causes the link BC to be put into a state of compression, so that an upward force acting along BC is balanced by an equal and opposite reaction F_1 of the pin B. The reaction F_1, as shown by the left-hand vector triangle, may be resolved into a force F acting tangentially to the path of the pin B, and a force F_2 acting along AB. No force passing through A has any turning moment about that point, and consequently the turning moment FR_1 is equal to PR_2. The forces

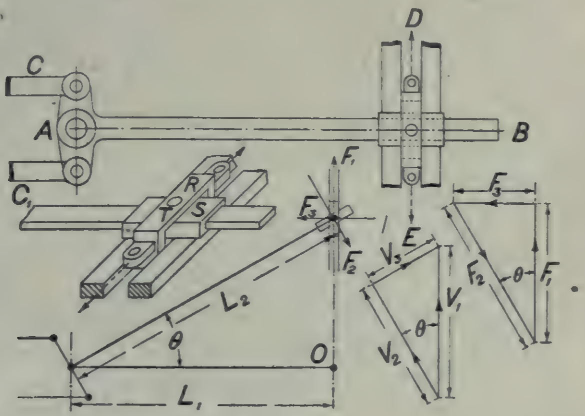

FIG. 108.

acting upon the pin C, viz., F_3, the reaction of the vertical guide surface, F_4, the resistance exerted upon the chisel and F_1, the downward force along BC, are represented by the right-hand vector figure.

Another example of a variable velocity ratio occurs in the device shown in Fig. 108. To a vertical post A, a three-armed lever is attached. Two links C and C_1 are pivoted to the ends of the short arms. The other ends of these links are attached to lever arms fixed to the rudder head of a ship. The third arm AB is a tiller, and upon this slides a sleeve S fitted above and below with pins T. The pins turn in a shackle R which slides between parallel guide-bars.

Along chains attached to R, pulls may be exerted in the direction D or E. Let F_1 be a pull along the chain in the direction D; if the tiller occupies the "fore-and-aft" position, represented by the line AB, a torque F_1L_1 is then transmitted to the rudder-head. Let the tiller move through an angle θ. Three forces now act upon the shackle R: F_1, the pull of the chain, F_2, a pressure exerted by the

pins T set up by the resistance to movement of the tiller, and F_3, a reaction of the guide surface. F_2 is equal to the force transmitted to the sleeve S in a direction at right-angles to the centre line of the tiller. The torque acting on the post A is now F_2L_2. When θ has any value exceeding nothing, F_2 exceeds F_1 and L_2 exceeds L_1; hence the torque in the second position of the tiller exceeds that in the first one. If the chain has a constant velocity V_1, V_2 is a component velocity of the pin T in a direction at right-angles to the tiller, and the other component velocity V_3 is the rate at which the sleeve slides along the tiller. The resultant of V_2 and V_3 is V_1.

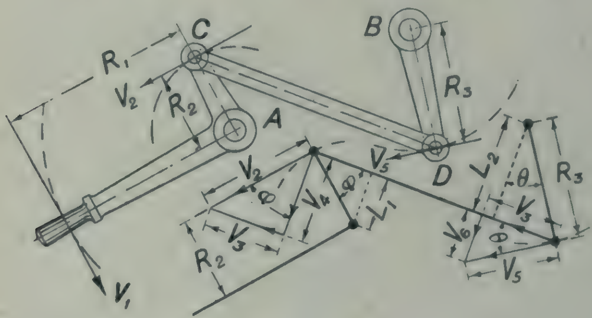

FIG. 109.

The rate of rotation or *angular velocity* of the tiller in the first position is $\frac{V_1}{L_1}$; in the second position it is $\frac{V_2}{L_2}$. The torque is F_1L_1 in the first and F_2L_2 in the second position. The rate at which energy is imparted to the tiller is the product of torque and angular velocity; in the first case it is $F_1L_1 \times \frac{V_1}{L_1}$ or F_1V_1, whilst in the second case it is $F_2L_2 \times \frac{V_2}{L_2}$ or F_2V_2. From the similarity of the two vector triangles we have

$$\frac{F_1}{F_2} = \frac{V_2}{V_1},$$

$$\therefore F_1V_1 = F_2V_2.$$

Hence if a constant force F_1 moves the shackle R at a constant speed V_1 energy is imparted to the tiller at a constant rate. As the angle θ increases, the torque on the rudder-head, transmitted from the post A, becomes greater whilst the rate of turning becomes correspondingly smaller. This contrivance is known as *Rapson's slide.*

The *Stanhope levers* (Fig. 109) afford another example of a mechanism involving a variable velocity ratio. At A is a gudgeon

pin and at B is a spindle, the position of each being fixed. A bell-crank lever turns about the pin at A. To the end of the short arm one end of a link CD is attached; the other end of the link is jointed to a lever arm BD keyed to the spindle at B.

A force applied to the handle and causing it to move with a velocity V_1 imparts a velocity V_2 to the pin C; it is manifest that $R_1 : R_2 :: V_1 : V_2$. Motion being transmitted through the link, a velocity V_5 is imparted to the pin D. Resolving the tangential velocity V_2 along and at right-angles to the link, component velocities V_3 and V_4 are obtained. Of these V_3 is transmitted to the pin D and forms one component of the actual velocity V_5 of that pin, the other component being V_6.

If the bell-crank lever is turned at a constant rate, V_2 remains unaltered, but the angles ϕ and θ diminish. In consequence, V_5 becomes continually smaller and therefore the spindle B rotates at a diminishing rate. In the given position, L_1 is the distance between the link and the pin at A, whilst L_2 is the distance between the link and the spindle at B. From similar triangles,

$$\frac{R_2}{L_1} = \frac{V_2}{V_3} \text{ and } \frac{R_3}{L_2} = \frac{V_5}{V_3}.$$

Therefore,

$$\frac{V_2}{R_2} = \frac{V_3}{L_1}, \text{ and } \frac{V_5}{R_3} = \frac{V_3}{L_2}.$$

$\frac{V_2}{R_2}$ is the angular velocity of either arm of the bell-crank lever and $\frac{V_5}{R_3}$ is the angular velocity of the lever BD. By substituting the equivalent values of these ratios,

$$\frac{\text{angular velocity of bell-crank lever}}{\text{angular velocity of lever BD}} = \frac{V_3}{L_1} \times \frac{L_2}{V_3} = \frac{L_2}{L_1}.$$

If the bell-crank lever is turned until ϕ becomes nothing, L_1 becomes nothing also and the above ratio will then be infinity.

If T is the tension in the link, the torque exerted at A is TL_1 and the torque exerted upon the spindle at B is TL_2. Therefore,

$$\frac{\text{torque at B}}{\text{torque at A}} = \frac{TL_2}{TL_1} = \frac{L_2}{L_1}.$$

As already shown,

$$\frac{\text{angular velocity at A}}{\text{angular velocity at B}} = \frac{L_2}{L_1}.$$

Hence,

$$\frac{\text{torque at B}}{\text{torque at A}} = \frac{\text{angular velocity at A}}{\text{angular velocity at B}},$$

from which it is evident that energy is received at B at the same rate as it is supplied at A.

Fig. 110 represents a *trip gear* for actuating the valve which controls the steam supply of an engine. A rod A connected to an eccentric produces a vibrating motion in the link B. A *tripping plate* is fixed to the curved piece C which is hinged to the link B. When the tripping plate moves in the direction K it forces down the end of the lever D which is released when the curved end of C is sufficiently deflected by passing over a roller E. The lever D is hinged to a spindle F of which the lower part is connected to the valve and the upper part to a compression spring.

When the lever D is released the compressed spring causes the

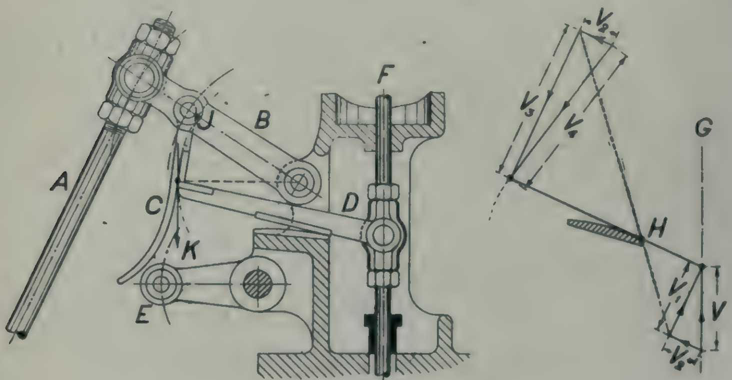

FIG. 110.

valve to close rapidly and cut off the steam supply. The point of cut-off is dependent upon the position of the roller E which is caused to rise or fall by the action of the governor. The lever D rests upon a curved plate. The position of the point of contact between the lever and the plate about which the lever rocks depends upon the inclination of the lever. When resting upon its seat there is a resultant downward steam pressure upon the valve which makes it advantageous to have a considerable lever ratio, but which is relieved when the valve is lifted.

In order to raise the valve rapidly a much smaller lever ratio becomes desirable as soon as the valve has left its seat. As the inclination of the lever decreases, the point of contact between the lever and the curved plate moves to the left and thus ensures the required result. The vertical line G represents the path of the right-hand end of the lever D. The path of the left-hand end is curved and is shown by a dotted line.

At any instant, let V be the speed of the valve spindle F. For V, substitute the component V_1 at right-angles to, and the

component V_2 along, the lever. V_2 is the velocity with which the lever slides upon the curved plate. A dotted line through H, the point of contact, enables the component velocity V_3 of the left-hand end of the lever to be found. V_3 acts in a direction at right-angles to the lever. Compounding V_3 with V_2 we obtain the actual velocity V_4 of the left-hand end of the lever along and tangential to its curved path. During the raising of the valve, the velocity V_4 varies only to a slight extent and determines V, since V_4 is the speed of that end of the lever to which motion is initially imparted. The velocity V_4 is shared by the lower end of the tripping plate on C. With regard to this plate, V_4 is the resultant of a tangential velocity in the direction K—equal to the tangential velocity of a point J in the link B—and a velocity which has a direction radial to the pin

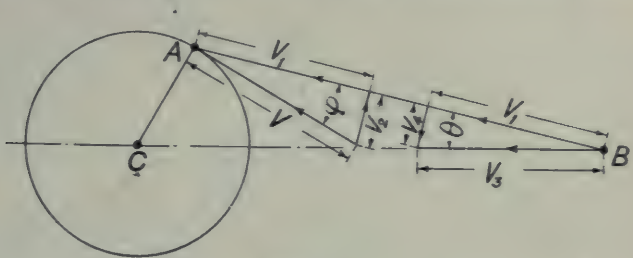

FIG. 111.

about which the link B turns. In order to avoid confusion these velocities are not represented in the vector figure.

Interchange of Rotary and Reciprocating Motion.—Let AB (Fig. 111) represent a connecting rod. The end A moves in a circle whilst the end B moves along a straight line CB. AC is a crank. If the point A has a tangential speed V, the point B will move along the line CB at a velocity V_3, which is related to V as follows: Resolving V along and at right-angles to the connecting rod, the component velocities V_1 and V_2 are obtained. V_2 represents a swinging motion of the end A about the end B, whilst V_1 is a velocity along the rod which is transmitted to B. Since BC, not BA, is the path of the point B, this point possesses a component velocity V_4 at right-angles to the direction of the rod. Hence V_1 being found, a vector to represent it may be set off from B and V_4 obtained: V_3 at once follows.

Without altering the position of A or the length AC, let the length AB be diminished. The angle ϕ will then become less, and V_1 in consequence becomes greater. Simultaneously, the angle θ becomes greater so that V_3 will increase even if V_1 did not. It is evident therefore that the relationship between the speed of a crank-pin and that of the crosshead connected to it is affected by

the ratio of the length of the crank to the length of the connecting rod.

Motion may be transferred from a connecting-rod to a rotating shaft by other means than a crank. In Fig. 112 A represents a spur-wheel keyed to a shaft whilst B is a wheel of equal size rigidly fixed to a connecting-rod. For the sake of simplicity, assume that the connecting-rod has no swaying motion but maintains a vertical position whilst the centre of the wheel B moves in a circle concentric with the axis of the shaft. A link, of which one end turns about the shaft and the other about a pin at the centre of B, ensures this action.

Let V_1 be the tangential velocity of the pitch circle of A, and V_2 the tangential velocity of the centre of the wheel B. The latter

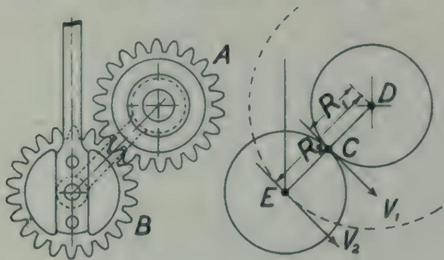

FIG. 112.

wheel is fixed to the rod, which has no turning motion; consequently B will have no turning motion either, notwithstanding the fact that its centre moves in a circle, and every point in the rod and the wheel B will share the velocity V_2. A tooth on the wheel B at the point C will therefore move at the velocity V_2 along the line marked V_1. Since the tooth on A situated at C moves instantaneously with a velocity V_1 along the same line and receives its motion from the tooth on B, it follows that V_1 equals V_2. R_1 being the radius of the pitch circle of either wheel, the angular velocity of the shaft is $\frac{V_1}{R_1}$.

Suppose that the wheels were replaced by a crank having a length R_2 which is equal to that of the link. Let the crank-pin E move at the same rate as before, viz., V_2. The angular velocity of the shaft then becomes $\frac{V_2}{R_2}$. It has been shown that V_1 is equal to V_2 and it is obvious that R_2 may be replaced by $2R_1$. Then

$$\frac{V_2}{R_2} = \frac{V_1}{2R_1}$$

The last ratio represents half the angular velocity imparted to the shaft by the wheel arrangement. In the case of the crank, the shaft

makes one revolution when the point E travels once around its circular path; therefore with the wheel arrangement the shaft makes two revolutions when E moves once around its path. Note, however, that this result is contingent upon the wheels being of equal size. The mechanism is called a *sun and planet motion*. It was applied to steam-engines by James Watt, but has long since been superseded by the crank.

In converting a reciprocating into a rotating motion it is not essential to employ a connecting rod. An instance is afforded by the device shown in Fig. 113. A circular casting A provided with a ring of teeth is bolted to a frame, concentrically with a crank shaft. The length of the crank is equal to half the radius of the pitch circle of the teeth on the ring A. A spur-wheel B, having a pitch circle

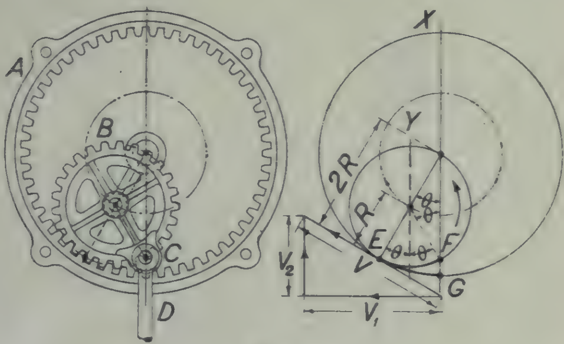

FIG. 113.

diameter equal to half that of the ring A, is free to turn on the crank-pin. A pin C is fixed to the wheel, its axis passing through the pitch circle of the wheel. A rod D is connected to the pin.

When the rod moves upward the wheel rolls around the ring and the pin C moves along a perfectly straight line intersecting the axis of the crank shaft. The diagram to the right of the figure is drawn in order to prove this assertion. X and Y are vertical lines. E is the point of contact of the two pitch circles. Assume that these circles are initially in contact at the point G, the pin C being situated at the same point. After the wheel has rolled as shown, let F be the point where its pitch circle cuts the line X. In any position, the smaller circle passes through the centre of the larger one. Lines being drawn from the centre of the wheel to E, F, and the centre of the crank-shaft, four angles, all equal and each marked θ, result.

The length of the arc of a circle subtending an angle at the centre of the circle is found by multiplying the length of the radius by the circular measure of the angle. Hence,

$$\text{arc EF} = R \times 2\theta,$$
$$\text{arc EG} = 2R \times \theta.$$

The arc EF has therefore the same length as the arc EG. Note that the point F was not defined as the position of the pin C, but as a point in the line X. Nevertheless F is the position of the pin C because the equality of the two arcs EF and EG results not merely from the geometrical relationships set forth but also from the rolling of the smaller pitch circle upon the larger one. As the conditions hold good for all possible positions of the pin C, its path must consequently be along the line X. Since the point F is always level with the point E, the speed of F at any instant is equal to the vertical component V_2 of the tangential velocity V of the point E.

The device is sometimes applied to printing machines; it was

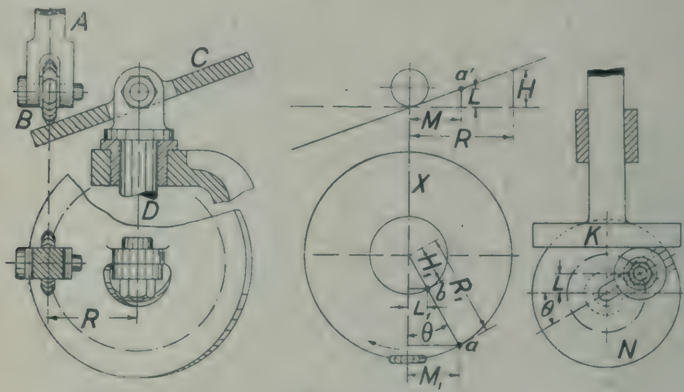

FIG. 114.

at one time used to obtain a straight-line motion for the piston-rod of a steam-engine, but is no longer used for that purpose.

When a rotating spindle is required to impart a reciprocating motion to a rod in the direction of the axis of the spindle the contrivance—known as a *swash-plate*—shown in Fig. 114 may be employed. The rod A is fitted with a small roller B which is always in contact with a plate C attached to the spindle D. The axis of the spindle is inclined to the plane of the plate. The line of contact between the roller and the plate is an ellipse, but the plan of the ellipse is a circle of radius R. The middle diagram shows the roller and the line of contact in plan and elevation, but the position of the roller with respect to the plate is not the same in this diagram as on the left.

Let the plate rotate about a vertical axis through the angle θ. The point on the plate aa^1 will then move a distance M to the left, and if the diameter of the roller were indefinitely small the roller and the rod would be raised exactly through the height L. After a quarter-turn of the spindle, the roller rises through a height H. Further turning of the spindle through a half-revolution results in

the roller descending through a height 2H, after which it ascends again. Concentric with the larger circle in the diagram, a smaller circle of radius H_1 is drawn. H_1 equals H, R_1 equals R, and M_1 equals M. From similar triangles,

$$\frac{H_1}{R_1} = \frac{L_1}{M_1} \text{ and } \frac{H}{R} = \frac{L}{M}.$$

$$\therefore L_1 = \frac{M_1 H_1}{R_1} \text{ and } L = \frac{MH}{R}.$$

Hence L_1 is equal to L.

L is the height the roller is raised by turning the spindle through the angle θ. If the point b were the position of a crank-pin at the end of a crank of length H_1, then when the crank turns from a position coincident with the line X through the angle θ, the crank-pin will move a distance L_1 to the right. This distance is equal to L, the height the roller is raised when the spindle turns through the angle θ. The result is true for any value of θ.

Suppose now that the roller on the end of the rod A is replaced by a rigidly attached cross-bar K, and that a roller turning about a pin bolted in a radial slot in a disk N is kept in constant contact with the cross-bar. Let the disk be attached concentrically to a rotating spindle and let the distance between the axis of the spindle and the roller centre be equal to H. This arrangement is equivalent to a crank which in turning through the angle θ will lift the rod through the height L. The kind of motion produced by a swash-plate is therefore identical with that produced by a crank working in the manner described. The crank has a variable length because the roller pin may be fastened at any point along the slot. A similar result is obtained in the case of the swash-plate by providing for a variable inclination between the plane of the plate and the axis of the spindle as indicated in the figure.

Cams.—A reciprocating motion may be imparted to a rod sliding in guides or to the end of a vibrating lever by a curved plate termed a *cam*. In Fig. 115 A is a cam keyed to and rotating with the shaft B for the purpose of operating a sliding rod D.

It will help the understanding of the relationship between the shape of a cam and the movement it causes if the following suggestions be acted upon. Shape to any smooth curve a thin piece of wood E. Choose any point F and from that point draw numerous radiating lines spaced at equal angles. Let these lines be numbered as shown, the zero line being the shortest that can be drawn from F. Pierce through the centre of a small circular disk G a conical hole. Upon a sheet of tracing paper H draw as many lines—parallel and equally spaced—as there are on the disk. At right-angles to these draw the line J. Upon a board draw a line KK_1, and by means of a drawing-pin, a bradawl, or a small screw, arrange that the curved

plate E may turn about the point F which is now fixed in the line KK_1.

Place the disk G in contact with the plate E, the centre of the disk and the zero line on the plate E both being situated upon the line KK_1. In this position mark the centre of the disk upon the board and through the point so found draw a line L at right-angles to the line KK_1. In the subsequent procedure the line J on the tracing paper is always to coincide with the line L. Turn the plate F around until the line numbered 1 lies along the line KK_1, place the tracing paper so that the line upon it numbered 1 also lies along the line KK_1, place the disk G in contact with the plate E and with its centre on the line KK_1; then mark the centre of the disk upon

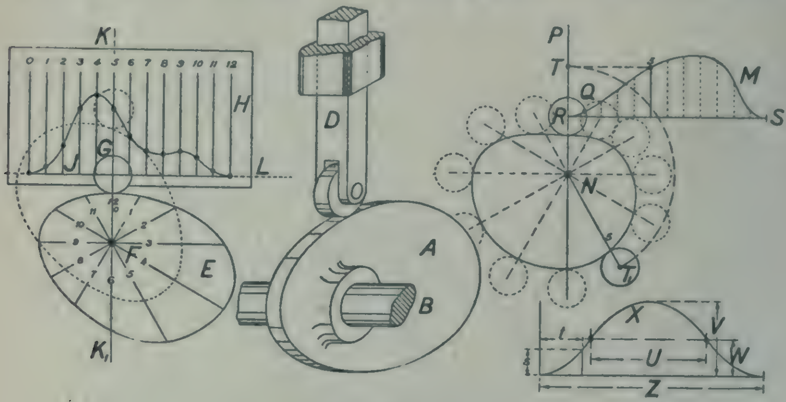

FIG. 115.

the tracing paper. Repeating these operations with respect to all the other lines and joining the points on the tracing paper, a curve is obtained of which the vertical ordinates represent the height that a rod provided with a roller of the same diameter as G would be raised by a cam of the same shape and size as E when rotating at a uniform speed. The horizontal dimensions of the curve are proportional to the time of movement, the distance between the vertical lines 0 and 12 being equivalent to the period of one revolution.

The procedure described may be reversed. Let the vertical ordinates of the curve M represent the displacement of a rod with respect to the time of turning of a cam. Any point N in the vertical line P may be chosen for the cam centre. A circle Q of any convenient diameter may represent the roller. RS is the intercept of the line of zero displacement representing the period of revolution of the cam. Divide RS into numerous equal parts and draw the vertical ordinates of the curve. Draw as many equally spaced lines radiating from the point N.

Consider any ordinate, the fifth, say. When the lift of the rod is equal to the height of this ordinate, the roller centre will be at T, the cam will have turned round far enough for the fifth radial line to coincide with the line P and there will be tangential contact between the roller and the cam. The distance NT which separates the centres of the cam and the roller is therefore transferred with a pair of compasses to the fifth radial line. With T_1 as centre, a circle of the same diameter as the roller is drawn. The operation being repeated until a circle has been drawn on each of the radial lines, a smooth curve touching all the circles tangentially gives the outline of the required cam.

As a rule, the time-displacement curve must be formed in accordance with definite requirements. Suppose, for example, that the rod is to be raised through the first half of its lift at a uniformly accelerated speed, that during the second half the speed is diminished at a uniform rate until the rod comes to rest, and that a similar sequence occurs during the descent of the rod. The curve X is formed in accordance with these requirements. The intercept Z represents the time of one revolution of the cam, U is half the length of Z, V is the height of the lift, and W is half the length of V. The portion of the curve above the intercept of length U is a parabola; the two parts below that intercept are portions of a parabola of the same shape. To draw these curves in accordance with given values of Z and V is a simple matter.

If during the first half of the lift the roller is raised a distance s in the interval of time t, then, a being the acceleration imparted to the rod,

$$s = \frac{at^2}{2} = kt^2,$$

because a constant acceleration was specified. A curve which passes through the upper ends of all such ordinates as s is a parabola. When the first half of the lift is accomplished the acceleration becomes negative and the time-displacement curve is modified accordingly. A cam shaped in conformity with the curve X causes during the lifting stroke a constant accelerating force to be exerted upon the mass of the rod when its speed is increasing, and requires a constant force in an opposite direction to accelerate the rod during the first half of the downward stroke. The latter result may be achieved by means of a spring or by using a cam of the form shown in Fig. 116 where a roller upon the end of a vibrating arm is made just to clear the sides of a groove in a rotating disk; as the direction of the accelerating force acting upon the roller changes, so the pressure changes from one side of the groove to the other.

If a high rate of reciprocation is required it is usually desirable to keep the accelerating force, exerted alternately by and upon the moving rod, at a constant amount, a result which is achieved by

shaping the cam in accordance with the curve X in Fig. 115. For a given rate of reciprocation, any smaller degree of acceleration at one part of the movement than the curve X involves must be balanced by a corresponding increase at another part so that the maximum accelerating force will then be greater than the constant force corresponding to the curve X. The accelerating force opposed by the inertia of the rod is additional to that which overcomes the external resistance imposed upon the rod.

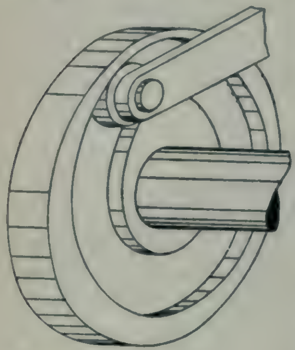

FIG. 116.

Velocity Curves.—The *pin and slotted lever* described in "Applied Mechanics—First Year" is reproduced in Fig. 117. The point A is the position of the crank-pin, and the vector of length V represents the tangential velocity of the crank-pin. Resolve the velocity V into two components, V_1 at right-angles to and V_2 along the direction of the lever. V_1 is a velocity imparted to the point in the lever where the pin is situated; V_2 is the rate at which the pin is at the same instant moving along the slot.

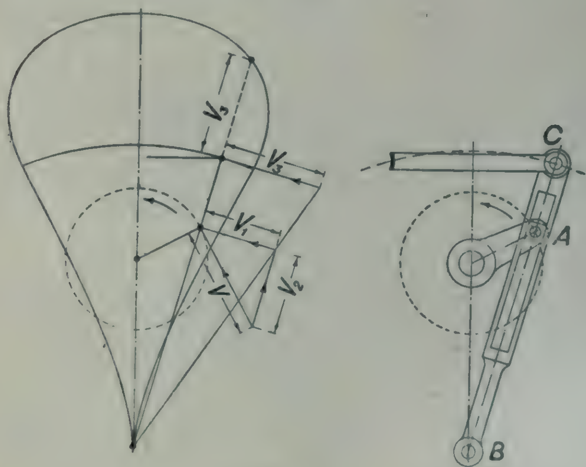

FIG. 117.

Since the lever turns about the pin B, any given point in it has a velocity, in a direction parallel to the vector V_1, which is proportional to the distance of that point from B. By means of a straight line passing through the turning point of the lever and the end of the vector V_1, it is possible to obtain a vector to represent

the velocity of any chosen point. In this manner the vector V_3 is obtained. Using as a base-line the circular arc described by the pin C at the end of the arm, an intercept equal to V_3 is marked off along the centre line of the lever. Repeating the construction for different positions of the lever, a number of similar intercepts are obtained. For the return movement of the lever, the intercepts are set off below the base-line. Joining the ends of the intercepts, a curve is obtained from which the tangential velocity of the pin C at any position of the lever may be found.

Instead of setting off the values of V_3 in the manner described they may be used to form vertical ordinates of a graph in which distance along the base represents the angle turned by the crank. It is scarcely necessary to point out that a diagram may thus be drawn to represent the relative velocities of two points in a mechanism for all possible relative positions of the parts, providing a method has been found which gives the relative velocities of the points for any one position.

EXAMPLES.

110. The leading screw of a lathe has 5 threads per in. The wheel on the mandrel has 40 teeth and gears with a wheel of 160 teeth on the leading screw. How many threads per in. will be cut on the work in the lathe?

111. The leading screw of a lathe has 3 threads per in. When cutting a screw with 8 threads per in. there are N teeth on the mandrel wheel, which gears with a 40-tooth wheel. On the same spindle as the latter and turning with it is a 30-tooth wheel which gears with a 50-tooth wheel fixed to the leading screw. What is the value of N?

112. A pantagraph reducing gear, made as in Fig. 95, is fitted to an engine having a piston stroke of 12 in.; the indicator diagram required is to be 3 in. long. The paths of the points B and C are parallel lines 18 in. apart. If the length CJ is 15 in., how long should EJ be?

113. Two parallel shafts with axes 2·4 in. apart are connected by an Oldham coupling (Fig. 99) and rotate each at a speed of 240 revs. per min. What is the speed of rubbing in ft. per min. of the feather against the side of the groove at the instant when the rubbing speed is the same for one feather as for the other?

114. The wheel of a bicycle is 28 in. in diameter. The sprocket-wheel on the same spindle as this has 12 teeth. The sprocket on the crank spindle has 48 teeth. How far will the rider move for 10 revs. of the crank spindle?

115. In Fig. 102 suppose that the wheels A, B and C are bevel-wheels of such form that B has 40 teeth, C has 20 teeth, and that the axis of A is so inclined that the three wheels are in mesh. What will then be the ratio of the angular speed of the spindle E to that of the rotating arm to which the handle D is attached ?

116. Assume that in Fig. 103 the wheels C and D have each 60 teeth and the pinions E and F each 20 teeth. When the car to which the mechanism is attached turns through the arc of a circle, each pinion rotates once about its axis whilst the wheel C makes one turn. The driving wheels are 4 ft. apart, C being connected to the one nearest the centre of curvature ; what is the radius of the curved path pursued by the driving wheel connected to C ?

117. The driving wheels of a vehicle are 4 ft. apart and each has a diameter of 3 ft. The wheels of the differential gear (Fig. 103) fixed to the axle have 50 teeth each ; the pinions connecting the wheels have 20 teeth each. If the inner driving wheel traces a circular arc of 6 ft. radius, what is the angle in radians between the initial and final positions of the driving axle for the period in which each pinion makes $\frac{1}{5}$ of a turn about the spindle ?

118. Let the wheels A and D of a Harrison winch (Fig. 105) each possess 12 teeth, and the wheels B and C each possess 14 teeth. How many times must the handle rotate in order to cause the drum to turn round once ?

119. In Fig. 106 assume that the sleeve G is moving along the shaft at a speed of 5 ft. per sec. at the moment that the toggles are inclined to the vertical at 10 degrees. At what speed do the pins P, P_1 move outward from the axis of the shaft ?

120. In Fig. 107 assume that the force P is 30 lb., that its line of action is 4 ft. distant from the pin A, that the arm AB is 4 in. long and at $45°$ to the vertical ; and that θ is an angle of $30°$. What force will then be exerted along the chisel E ?

121. In Fig. 108 let L_1 be a distance of 6 ft. If the chains attached to the shackle R move at a speed of 1 ft. per sec. whilst the tiller is at an angle of $30°$ to the centre line of the vessel (i.e. $\theta = 30°$) at what speed does the slide S move along the tiller (*a*) ; what is the angular velocity of the tiller in radians per sec. (*b*) ; and what torque is exerted in lb.-ft. about the vertical post A if the pull in the chain amounts to 500 lb. (*c*) ?

122. In Fig. 109 let R_1 be 12 in., R_2 be 3 in., and R_3 be 4 in. A force of 20 lb. being applied at the end of the lever of length R_1 and the angles ϕ and θ each being $30°$, what torque in lb.-in. is exerted upon the pin B ?

123. A crank 1 ft. long is turned round by a connecting rod 3·5 ft. long. If the crank-pin has a uniform tangential speed of 100 ft. per sec., what is the speed of the crosshead-pin when

the crank and the connecting rod are mutually at right angles?

124. In Fig. 113 assume that at a particular instant the speed of the point of contact between the pitch circles is equal to the speed of the pin C; what fraction of its journey will the pin then have completed?

125. In Fig. 113 assume that the angle θ is $60°$; if the force acting along the vertical rod D is then 200 lb., what force is tending to shear the crank-pin about which the toothed wheel turns?

126. In the mechanism shown in Fig. 113, suppose that the pin C has completed one-fourth of its vertical movement; what is then the ratio of the speed of the point of contact between the pitch circles to the speed of the pin C?

127. A rod is raised and lowered by a swash-plate (Fig. 114). The distance between the axis of the rod and the axis of rotation of the swash-plate is 4 in. If the total lift of the rod is 2·91 in., at what angle is the axis of rotation inclined to the plane of the swash-plate?

128. A swash-plate imparts a lift of 2 in. to a rod of which the centre line is 4 in. distant from the axis of rotation of the swash-plate spindle. Starting from its lowest position, the rod rises 0·5 in.; through what angle has the swash-plate turned?

129. The crank which operates a pin and slotted-lever mechanism (Fig. 117) is 6 in. long. If the return stroke of the cutting tool is to take place in half the time of the working stroke, how far must the gudgeon-pin B be fixed from the axis of rotation of the crank?

130. A pin and slotted-lever mechanism (Fig. 117) is operated by a crank 5 in. long. The distance from the pin B to the centre of the crank shaft is 15 in.; the length BC is 25 in. When the crank makes 100 revs. per min. what are the highest and lowest speeds, in ft. per sec., of the pin C?

131. A cam is required to raise a rod through a distance of 3 in. with a uniform acceleration which is positive during the first half of the lift and negative during the second half. During the descent the same acceleration is produced, negative for the first half and positive for the second. Allowing 4 in. between the centre of the cam-shaft and the centre of the roller—the latter having a diameter of 2 in.—set out the shape of the cam.

132. A circular plate of 6 in. diameter is mounted eccentrically upon a spindle, the centre of the spindle being 1·25 in. distant from the centre of the plate. The plate is used as a cam to cause reciprocating motion in a rod sliding in guides. At the end of the rod is a roller 2 in. in diameter. The centre line of the rod passes through the centre of the spindle. Draw a curve of time-displacement, the period of revolution of the cam being represented by a horizontal intercept 6 in. long.

CHAPTER VII

The Pendulum.—A pendulum swings to and fro with a *periodic motion ;* this term applies to any instance in which a point repeatedly occupies the same position and is moving in the same direction at equally separated intervals of time. In travelling from one extreme position and back to the same place a pendulum performs one *complete vibration*, or *oscillation*. The time in which this movement occurs is the *period of vibration*, or *oscillation.*

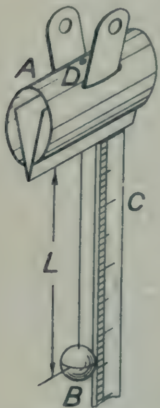

FIG. 118.

There are few people who have not noticed that a long pendulum vibrates at a slower rate than a short one. Let our object be to discover what relationship exists between the length of a pendulum and its period of vibration. In Fig. 118, A represents a steel letter-clip which may be screwed to a vertical post so as to support by means of a fine silk thread a small metal ball B. By drawing the thread through a small hole at D the length L of the pendulum may be adjusted with the aid of a scale C ; to facilitate this adjustment, a line is engraved around the ball.

Drawing the ball—which is termed the "bob" of the pendulum—aside, vibration will commence when it is released and will continue for a considerable length of time. If all the vibrating material could be concentrated in a ball of indefinitely small size we should be able to realise the abstract conception known as a *simple pendulum ;* a fair approximation is obtained by using a thin thread and a small ball made of a dense material.

Measure the length L in ft. and count the number of *complete* vibrations made in exactly one minute. Dividing 60 sec. by the number of vibrations, the period of one vibration is obtained. The following table contains the period of vibration T for simple pendulums of different lengths, and was obtained in the manner set forth.

Length (ft.)	0·2085	0·333	0·50	0·667	0·833	1·00
Number of vibrations per min.	117	94	77	67	60	54
Period, T sec.	0·513	0·64	0·78	0·896	1·00	1·11
T^2	0·263	0·41	0·608	0·804	1·00	1·233

Length (ft.) . . .	1·166	1·333	1·50	1·666	1·833
Number of vibrations per min.	50	47	44	42	40
Period, T sec. . . .	1·2	1·276	1·363	1·43	1·5
T^2	1·44	1·63	1·86	2·042	2·25

The table includes the square of the number of seconds in each period, for when the period is plotted against the length of the pendulum, as in Fig. 119, the resulting graph A resembles Fig. 6.

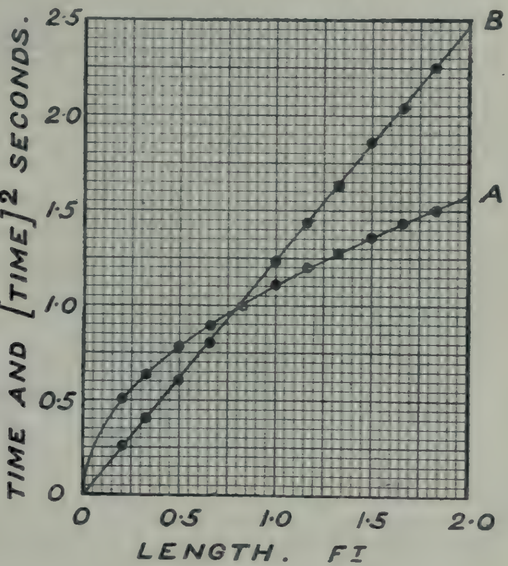

FIG. 119.

Pursuing the same course as in the case illustrated by Fig. 6, the graph B, having a slope of 1·23, is obtained. T being the number of seconds occupied by a pendulum L ft. long in performing one vibration, the equation to the graph is therefore

$$T^2 = 1{\cdot}23L$$

or

$$T = \sqrt{1{\cdot}23L} = 1{\cdot}11\sqrt{L}.$$

In each semi-vibration it is noticeable that the bob commences its journey from a state of momentary rest, travels at an increasing speed until it has gone half-way, and then slows down to a state of momentary rest before commencing the return journey. The following experiment reveals the relationship between the forces acting on the pendulum bob.

Suspend a heavy piece of metal of W lb. weight upon a long thin steel wire. At a point P in this wire which is situated a little above the suspended metal, attach a horizontal cord and let it pass over a grooved pulley A, Fig. 120. Fix a length scale B horizontally, just behind the plane of the vertical wire and the horizontal cord. Observe the reading of this scale when the wire supporting the metal hangs in a vertical line. Attach a small weight C to the end of the cord. The suspended metal is now drawn aside and the supporting wire becomes inclined. The horizontal component of the upward

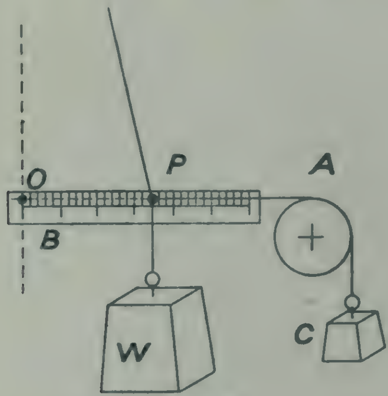

FIG. 120.

pull along the sloping wire is balanced by the pull produced in the horizontal cord by the weight C. Observe and record the reading of the scale together with the horizontal pull due to the weight C. The displacement of the point P from the point O is deduced from the scale readings. Increasing the load C the procedure may be repeated a number of times. In this manner the following numbers were obtained; plotting them, the graph Fig. 121 resulted.

Weight of metal supported = 14 lb.

Length of supporting wire from the point of suspension to the point of attachment of the cord = 4·962 ft.

Displacing force (lb.)	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
Displacement (ft.) .	0·036	0·071	0·107	0·142	0·177	0·213	0·251	0·288	0·321

The graph indicates that the force pulling the piece of metal aside is proportional to the displacement of the metal from the point O.

Simple Harmonic Motion.—Let the horizontal cord be removed, and the heavy piece of metal be so attached to the vertical wire that

its centre of gravity is, if at all, slightly above the point P (Fig. 120). Allow the metal to vibrate as a pendulum. The horizontal accelerating force at each position of the point P is now exactly equal to the force previously exerted along the horizontal cord ; it is, in fact, the same force that was previously balanced by the pull along the cord. Extend the line of the cord in Fig. 120 to meet the vertical line in the point O.

Since the acceleration of a constant mass is proportional to the force causing it, and because the horizontal accelerating force is proportional to the displacement of the metal, the acceleration of the point

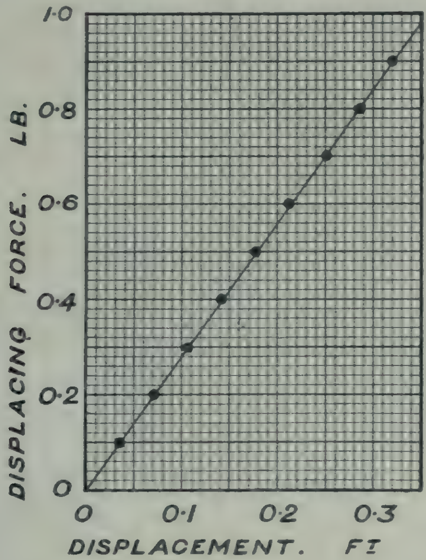

FIG. 121.

P toward the point O is at every instant proportional to the displacement of P from O.

The resulting movement of the point P is termed *simple harmonic motion*. For the sake of brevity this is often represented by the initial letters, S.H.M. When the moving point reaches O the greatest speed has been attained, but the acceleration is reduced to nothing. After passing O the acceleration becomes negative, that is, instead of acting from right to left it acts from left to right ; in consequence, the speed diminishes to nothing, after which the persistence of the negative acceleration causes the point to return to O. A reversal of the direction of the acceleration follows and the moving point comes to rest for an instant in its initial position.

It is easy to find examples of S.H.M. other than that afforded by a pendulum ; the succeeding remarks are accordingly stated in

a general manner, although their meaning may more easily be grasped by considering them in relation to the foregoing experiment.

In Fig. 122, PN represents the path of an object weighing W lb. and oscillating between the points P and N with a S.H.M. The distance OP or ON is termed the *amplitude* of the oscillation. Displacements are measured along the horizontal line, O being the zero, and P the position of maximum displacement. The sloping line is a graph of which the vertical ordinates represent values of

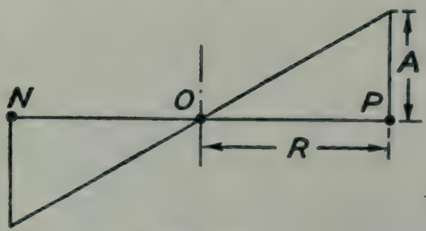

FIG. 122.

the acceleration. Let F lb. be the horizontal force urging the object toward O when the displacement is R. The force F diminishes uniformly; by the time the point O is reached F equals nothing and its average value over the space R is $\frac{1}{2}$F. Applying this result to the case of the heavy vibrating piece of metal shown in Fig. 120, the work done on the metal during a quarter-vibration is $\frac{1}{2}F \times R$ ft.-lb. In consequence, the metal is endowed with K.E., which, if its speed at O is V ft. per sec., is equal to $\frac{WV^2}{2g}$ ft.-lb. Hence,

$$\text{energy acquired} = \text{work performed}$$

$$\frac{WV^2}{2g} = \frac{FR}{2}.$$

If a, represented by the ordinate A in Fig. 122, be the acceleration produced in the metal by the action of the force F,

$$F = \frac{W}{g}a.$$

Putting this expression for F in the previous equation, we have

$$\frac{WV^2}{2g} = \frac{WaR}{2g}.$$

From this it follows that:

$$a = \frac{V^2}{R}$$

Although F is dependent upon the weight of the metal, yet, as the final equation shows, the acceleration a is not. The last equation is important and will be applied in dealing with the next example.

Fig. 123 indicates the construction of a steam-pump. A is the steam-piston, B is the water-piston. The pressure of the steam

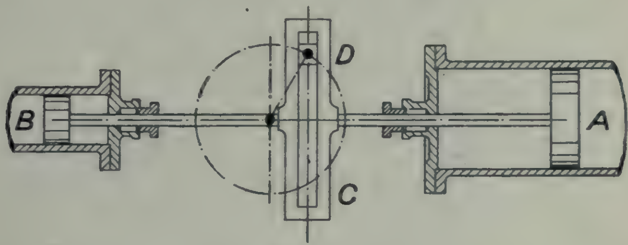

FIG. 123.

upon A balances the resistance of the water exerted upon B. A frame C having a deep vertical slot is fixed between the rods attached respectively to the steam-piston and the water-piston. A block of metal slides in this slot; in the centre of the block is fitted a crank-pin of which the centre moves around the dotted circle. The crank-pin is used to connect the reciprocating parts to a fly-wheel.

Assume that the fly-wheel rotates at a uniform speed of 60

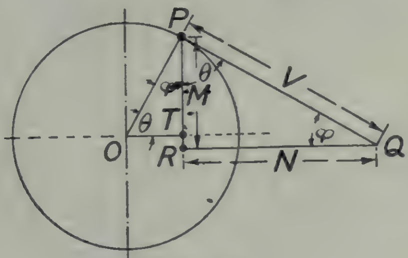

FIG. 124.

revs. per min. and that the crank is 3 in. long. The speed of the crank-pin will then be $\frac{2\pi \times 3 \text{ in.} \times 60 \text{ r.p.m.}}{60 \times 12}$ ft. per sec., or $\frac{1}{2}\pi$ ft. per sec. This speed is represented by the vector PQ in Fig. 124.

The circle in Fig. 124 represents the path of the crank-pin and P is a point coinciding with any position occupied momentarily by the crank-pin. Resolving the tangential speed V into components, we have N representing the horizontal velocity and M the vertical velocity of the crank-pin at the position P. Only the simplest of geometrical reasoning is required to show that the angles marked

θ and ϕ are respectively equal; hence the triangles QRP and PTO are similar. For any position of the point P, therefore,

$$\frac{N}{V} = \frac{PT}{OP}.$$

Let the radius of the circle be so chosen as to represent V to a scale of speed. In accordance with the last equation, the vertical intercept between any point in the circle and the horizontal diameter must then represent, to the same speed scale, the horizontal velocity of P at the point where the intercept is drawn. In other words, if the radius of the circle is equal to V—which is the length of a vector representing tangential speed—the circle may be regarded as a

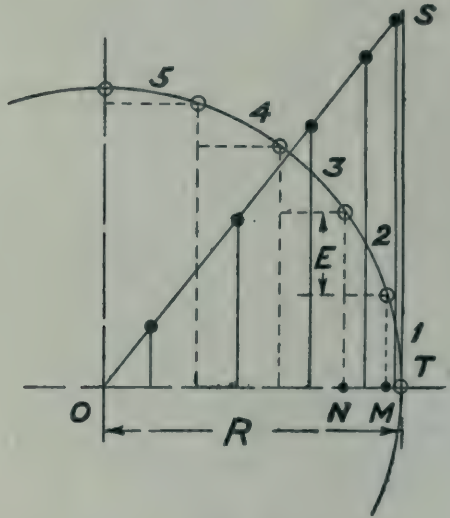

FIG. 125.

graph of which the vertical ordinates show the horizontal velocity of P with respect to the position of P. Applying this to the example under consideration, let the scale of velocity be 0·2 ft. per sec. to 1 cm. and the radius required will be $\frac{\pi}{2 \times 0{\cdot}2}$ cm. or 7·85 cm.

Regard the circle in Fig. 125 as having this radius. Divide one quarter of the circle, as shown, into, say, five equal parts. One revolution being performed in a second, the time occupied by the crank-pin in moving through any one of these arcs is $\frac{1}{5} \times \frac{1}{4} \times 1$ sec. or 0·05 sec. During the movement through the second arc the horizontal velocity increases by the amount E, which by means of the scale of velocity is found to represent 0·434 ft. per sec. Since

this increase takes place in 0·05 sec., the average acceleration or increase of velocity per sec. during the interval is $\frac{0 \cdot 434 \text{ ft. per sec.}}{0 \cdot 05 \text{ sec.}}$ or 8·68 ft. per sec. per second. In this manner the average acceleration during each of the five intervals may be found. The diagram in Fig. 125 yielded the following results when treated in the way described :—

Interval	1st	2nd	3rd	4th	5th
Increase of velocity (ft. per sec.)	0·484	0·434	0·344	0·22	0·08
Average acceleration (ft. per sec. per sec.)	9·68	8·68	6·88	4·4	1·6

Whilst the crank-pin moves through the second arc, the movement of the reciprocating parts is from M to N. Assume that the average acceleration of these parts over the space MN, which is shown in the table above to be 8·68 ft. per sec. per second, agrees with the actual acceleration at a point midway between M and N. At this point set up a vertical ordinate representing to scale an acceleration of 8·68 ft. per sec. per second. Treat all the other intervals in like manner and draw a line through the upper ends of the ordinates. This line, OS, proves to be perfectly straight. Now draw the ordinate ST tangential to the circle and find what acceleration it represents. The result obtained is 9·9 ft. per sec. per second.

Compare Figs. 125 and 122. It is evident that if the assumption respecting the average acceleration is correct, the reciprocating parts of the pump possess a simple harmonic motion and ST represents the horizontal acceleration at the commencement of the stroke. At mid-stroke, which corresponds to the zero displacement point, the horizontal velocity of the reciprocating parts coincides with V, the tangential speed of the crank-pin, for at this point the vertical velocity has diminished to nothing. Hence if R ft. be the radius of the crank-pin circle the acceleration represented by ST is equal to $\frac{V^2}{R}$ (see p. 168). As R is 0·25 ft. and V is $\frac{1}{2}\pi$ ft. per sec.,

$$\frac{V^2}{R} = \frac{(0 \cdot 5\pi)^2}{0 \cdot 25} = 9 \cdot 86 \text{ ft. per sec. per second.}$$

This numerical result agrees closely with the value found graphically, viz., 9·9 ft. per sec. per second. When the matter is treated from another standpoint, as it will be later, it will be seen that the method is fully justified and that the reciprocating parts actually do possess a simple harmonic motion.

Period of Vibration.—It is now a simple matter to determine the time of vibration of an object possessing S.H.M. Referring to Fig. 123 it is plain that during one revolution the crank-pin moves a

distance of $2\pi R$ ft. whilst the reciprocating parts move $4R$ ft. Let T sec. denote the time in which these movements are simultaneously performed. Then, as V denotes the uniform speed of the crank-pin,

$$2\pi R = VT$$

or

$$T = 2\pi \times \frac{R}{V} \ldots\ldots\ldots\ldots (a).$$

Consider the displacement of the crank-pin and the acceleration to which it is subjected at any chosen point. Referring to Fig. 122 we see that

$$R : \frac{V^2}{R} :: \text{displacement} : \text{acceleration}.$$

Hence,

$$\frac{R^2}{V^2} = \frac{\text{displacement}}{\text{acceleration}}$$

and

$$\frac{R}{V} = \sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

By substitution in the equation (a)

$$T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

The result may be checked by applying it to the case of the simple pendulum. From the graph (Fig. 121) it is found that when the displacement of the metal is 0·25 ft. the displacing force F is 0·7 lb. The weight W of the metal in this experiment was 14 lb.; let a be the acceleration which the displacing force F would cause, then from the fundamental equation, $F = \frac{W}{g}a$,

$$a = \frac{Fg}{W} = \frac{0 \cdot 7 \times 32 \cdot 2}{14} = 1 \cdot 61 \text{ ft. per sec. per second.}$$

If the metal is allowed to oscillate, T being the period of a vibration and the displacement D being 0·25 ft. as given above,

$$T = 2\pi \sqrt{\frac{D}{a}} = 2\pi \sqrt{\frac{0 \cdot 25}{1 \cdot 61}} = 2 \cdot 47 \text{ sec.}$$

On p. 165 is stated the relationship between the length of the pendulum and the time of a vibration as found experimentally, viz.,

$$T = 1 \cdot 1 \sqrt{L}.$$

Referring to p. 166 we find that the wire supporting the 14 lb. weight is 4·962 ft. long, so that

$$T = 1\cdot 1\ \sqrt{4\cdot 962} = 2\cdot 45 \text{ sec.}$$

From the close agreement between the two values of T we may feel assured that our equations are true expressions of actual fact.

The slope of the graph in Fig. 119 may now be determined rationally. In Fig. 126 P represents the bob of a simple pendulum of length L ft. W lb. is the weight of the bob, which requires a force of F lb. to cause a displacement of D ft. To the right of the figure is a triangle representing the balanced forces acting upon the bob. From the manifest similarity of the triangles in the figure:

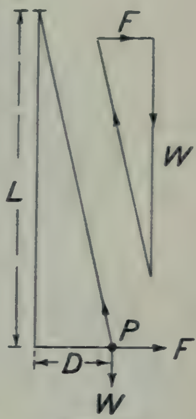

FIG. 126.

$$\frac{L}{D} = \frac{W}{F}.$$

The force F may be expressed in terms of the acceleration it will produce in an object weighing W lb. Thus:

$$F = \frac{W}{g}a.$$

The previous equation may now be restated as follows :—

$$\frac{L}{D} = \frac{Wg}{Wa} = \frac{g}{a}.$$

Hence,

$$\frac{D}{a} = \frac{L}{g}.$$

The symbols D and a have exactly the same meaning as in the equation given on p. 172 and in consequence,

$$T = 2\pi\sqrt{\frac{D}{a}} = 2\pi\sqrt{\frac{L}{g}} = \frac{2\pi}{\sqrt{g}}\sqrt{L}.$$

Compare this with the equation to the graph A (Fig. 119), viz..

$$T = 1\cdot 11\ \sqrt{L}.$$

We have only to calculate the numerical value of $\frac{2\pi}{\sqrt{g}}$ and if this agrees with 1·11 no reasonable doubt can exist as to the correctness of our conclusions.

Vibration of a Spring.—If an object be suspended by a helical spring, as A (Fig. 127), forced downward into the position B and then released, it will rise to a position higher than that of A. After this it will oscillate up and down until it gradually comes to rest. A piece of iron weighing 14 lb. being suspended in this manner was found to make 190 complete oscillations in 2 min. Hence the time of vibration was 0·632 sec. Removing the metal, applying successive loads and measuring the stretch produced by each, the rate of extension of the spring was found by means of a graph to be 42·5 lb. per ft. of extension.

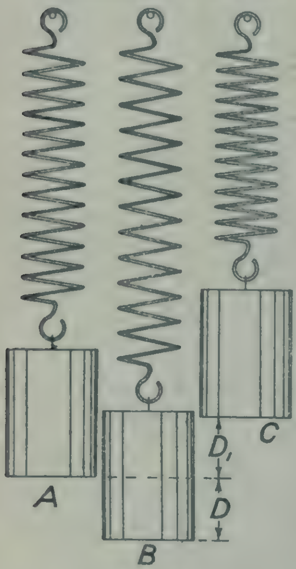

FIG. 127.

These data are represented graphically in Fig. 128. The straight line MN is the graph obtained by plotting load on the spring against the extension of the spring. The point P in the graph corresponds to an extension E due to the weight W of the suspended metal. O corresponds to the stationary position of the metal. Upon applying a force F, a downward displacement from O, equal to D, takes place. The tension in the spring is then W + F lb. Upon the removal of the downward thrust, the metal rises until a displacement D_1, equal to D, above the position O has occurred. The tension in the spring is then W — F_1, F_1 being equal to F. Comparing the diagram with Fig. 122 it is plain that a simple harmonic motion will be imparted to the metal supported by the spring.

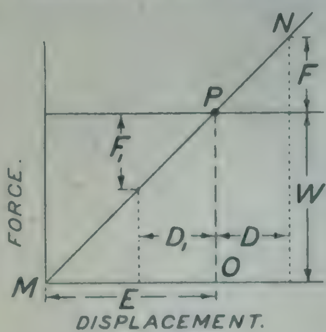

FIG. 128.

Let F lb. be the force which the spring would exert if extended 1 ft. F, it has already been stated, is equal to 42·5 lb. As a matter of fact, the greatest extension was less than 2 in., but what we require is the *ratio* between displacement and acceleration. The force F lb. acting upon $\frac{W}{g}$ mass units imparts an acceleration of a ft. per sec. per second.

When the displacement is 1 ft., F equals 42·5 lb., and since W is 14 lb.,

$$\text{acceleration} = a = \frac{Fg}{W} = \frac{42{\cdot}5 \times 32{\cdot}2}{14} = 97{\cdot}7 \text{ ft. per sec. per second.}$$

Let T be the time in seconds of an oscillation. Since the displacement D is 1 ft.,

$$T \doteq 2\pi\sqrt{\frac{D}{a}} = 2\pi\sqrt{\frac{1}{97{\cdot}7}} = 0{\cdot}635 \text{ sec.}$$

This agrees with the observed period of oscillation, viz., 0·632 sec.

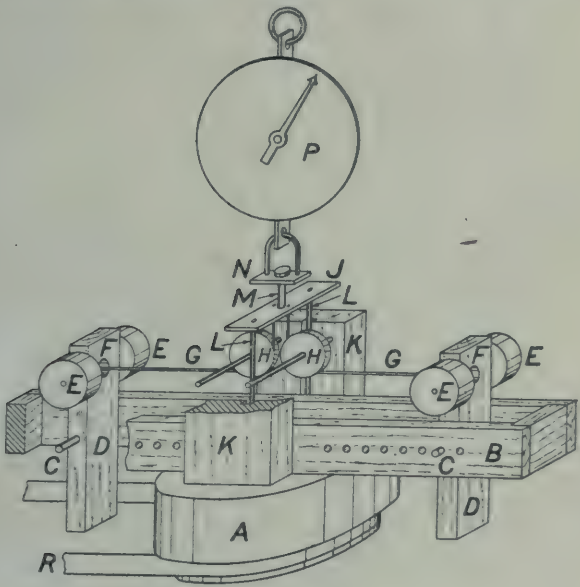

FIG. 129.

Centrifugal Force.—It is common knowledge that when a heavy object is moved along a curved path a force must be exerted upon the object in a direction normal to the curve in order to overcome the tendency of the object to maintain its motion in a straight line. Travelling in a vehicle at a fair speed around a sharp curve, one's bodily sensations enable this fact to be realised. It is important to ascertain exact relationships between the radius of the curved path and the speed and mass of the whirling object.

Fig. 129 represents an apparatus devised and used by the writer for determining these relationships. A circular disk A is attached firmly to a pulley of 6 in. diameter. The pulley is free to turn

about a vertical gudgeon pin provided with a collar at its upper end. A rectangular frame B is screwed to the disk A, so that when the pulley turns the frame swings around the axis of rotation of the pulley. A row of holes, each $\frac{1}{4}$ in. in diameter, is made in each side of the frame. The centre line of each hole is placed at a distance from the axis of rotation equal to some multiple of 0·1 ft. Through any pair of holes equidistant from the axis of rotation, spindles CC are passed; upon each of these a rectangular piece of wood D is mounted. Each of the pieces DD is balanced upon its spindle and turns about it with the least possible resistance.

Four small metal disks of equal weight, each marked E, are firmly fixed to the pieces marked D, two to each of them. The object of the apparatus is to measure the radial pull required to cause these pieces of metal to maintain a circular path of a given radius when moving at a known speed. In order to vary the amount of material experimented upon, each of the pins passing through the centres of the disks is long enough to attach about ten disks—weighing, say, 0·1 lb. each—to the frame. A hole F is bored through each piece D so that a cord G may be tied to the pin. Each cord is deflected by means of a small grooved pulley H into a vertical direction. The upper ends of the cords are fixed to a plate J. The grooved pulleys are mounted upon spindles carried by brackets KK screwed to the frame. In each of these brackets there is a groove which permits a rod L riveted to the plate J to move freely along it. The object of these rods is to prevent the vertical parts of the cords becoming entwined.

A stem M, connected to the plate J, is provided with a head which is capable of transmitting any pull along the cords to the plate N. When the apparatus is in operation the stem M and every part of the apparatus below it are rotating about a vertical axis. The plate N is used to connect the rotating parts to a spring-balance P. The balance was suspended from a pulley-tackle so that it might be raised or lowered. Rotation was effected by connecting the driving belt R with a lathe. Partly by means of the speed cones of the lathe, partly by turning down a wooden cylinder, running between the lathe centres, which served the purpose of a driving pulley, changes of speed were effected.

In the first place, the total weight of the parts marked N, J, and L was found. This weight was subtracted from all subsequent indications of the balance in order to find the sum of the pulls induced in the cords by the action of all the rotating parts marked E. The parts marked D did not influence the reading of the spring-balance, since each of them was balanced about the spindle C.

In the first set of experiments the spindles CC were adjusted at a distance of 0·9 ft. from the axis of rotation. The speed was maintained constant at a rate of 138 revs. per min. This was the highest speed attained at any time, so that no difficulty occurred in counting the number of revolutions. Any such difficulty may be

reduced by colouring brightly one of the parts D. Before the counting was commenced the balance was adjusted to a height which caused each of the parts D to become vertical. As the common centre of gravity of the two cylinders E fixed upon each piece D is in the same vertical line as the centre of the spindle C, each of these common centres of gravity rotated in a circle of 0·9 ft. radius. After ascertaining the radial pulls induced along the cords, the weights of the disks E were altered and the observations

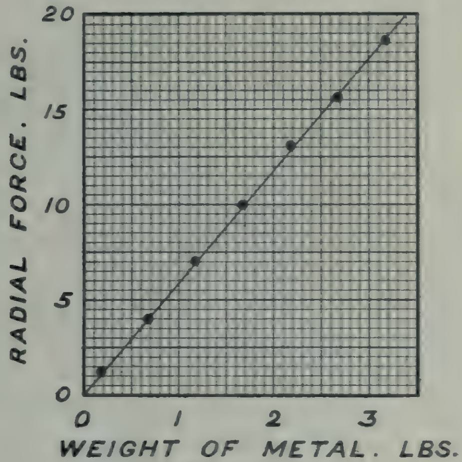

FIG. 130.

were repeated. The following table was obtained after successive repetitions of these operations :—

Total weight of cylinders E (lb.)	0·19	0·69	1·19	1·69	2·19	2·69	3·19
Total radial force (lb.)	1·2	4·0	7·0	10·0	13·2	15·7	18·6

Plotting the radial force against the weight of metal in the disks E, the graph (Fig. 130) resulted. This is straight and passes through the origin. Hence,

When a heavy object moves at a constant speed along a circular path, a radial force proportional to the weight of the object must be applied to keep the object in its circular path.

The equation to the graph is

$$\text{radial force (lb.)} = 5{\cdot}86 \times \text{weight of rotating body (lb.)}$$

The radial force acts in a direction at right-angles to that of the weight of the rotating disks ; consequently, one force cannot

influence the other. Moreover, the weight of each disk is balanced by an upward reaction of the spindle C upon which the bar D is pivoted. We must conclude therefore that the radial force is dependent not upon the *weight* but the *mass* of the rotating disks.

Throughout the next set of experiments the speed was 131 revs. per min., and the total weight of the rotating disks E was 2·69 lb. The radius of the centre of gravity of each pair of disks was altered as shown below.

Radius of path of disks E (ft.)	0·4	0·5	0·6	0·7	0·8	0·9	1·0	1·1
Total radial force (lb.)	6·3	8·0	9·6	11·1	13·0	14·2	16·1	17·7

Plotting the radial force against the radius of the centre of gravity,

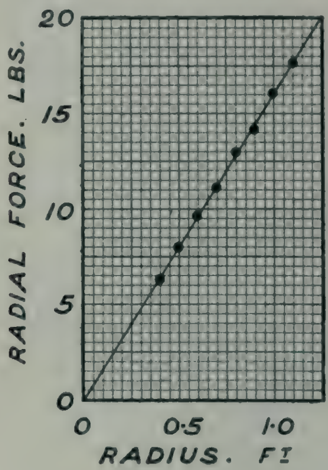

FIG. 131.

the graph (Fig. 131) was obtained. This also is straight and passes through the origin ; in consequence,

When a heavy object moves around a circular path at a constant rotational speed, the radial force required to keep it in its path is proportional to the radius of the path.

Observe that it is the *rotational* speed, not the *tangential* speed, which is constant. The equation to the last graph is

total radial force (lb.) = 16 × radius of the path of the C of G. (ft.).

In the last set of experiments neither the weight of the rotating metal nor the radius of the path of the centre of gravity was altered. The speed of rotation was repeatedly changed and at each speed the radial force was determined. The observations were as follows :—

Radial force (lb.) .	1·7	3·8	6·2	7·7	11·2	12	13·7	16·2	17·5	18·7
Revs. per min. . .	42	63	79	89	106	111	118	129	133	138

Plotting the radial force against the speed of rotation gives the

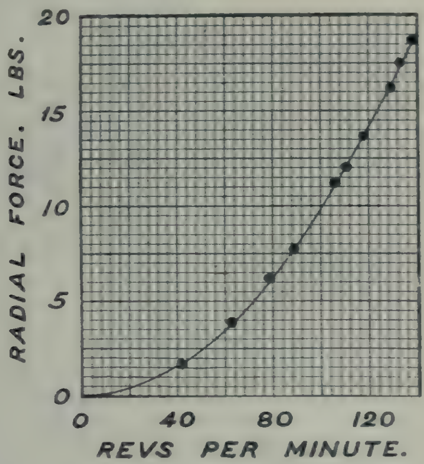

FIG. 132.

curved graph (Fig. 132) of which the shape is suggestive of a parabola. Should this actually be the case,

$$\text{radial force (lb.)} = M[\text{revs. per min}]^2.$$

and

$$\frac{\text{radial force (lb.)}}{[\text{revs. per min.}]^2} = M.$$

That is, the number representing the radial force bears a constant ratio M to the number representing the square of the rotational speed. In consequence a straight-line graph of which the slope is M will result if the radial force is plotted against the square of the rotational speed. In the following table, F represents the radial force as given in the last table and N^2 represents the square of the corresponding rotational speed.

F (lb.) .	1·7	3·8	6·2	7·7	11·2	12·0	13·7	16·2	17·5	18·7
N^2 .	1765	3970	6240	7920	11240	12300	13920	16650	17700	19000

The values of N^2 were obtained by the use of a slide-rule, and are therefore merely approximations. Fig. 133 is a graph plotted

from the last sets of numbers, and as this proves to be straight, our suppositions are justified. The equation is

$$\text{radial force (lb.)} = 0{\cdot}000985\ [\text{revs. per min.}]^2.$$

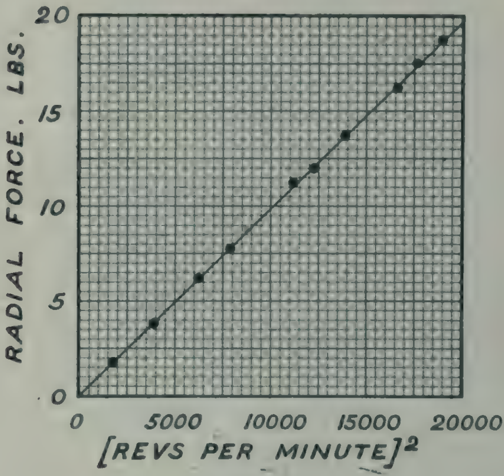

FIG. 133.

The equation applies also to Fig. 132.

Expressing the result in a general form, we have .

When a heavy object moves along a circular path the radial force required to keep it in its path increases proportionately to the square of the rotational speed.

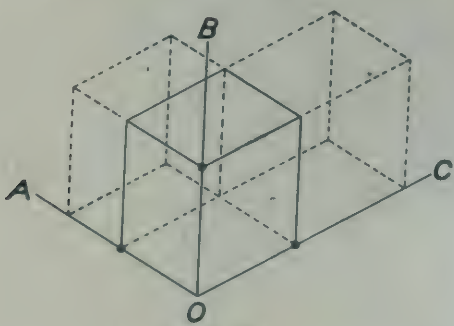

FIG. 134.

The combined results of all the experiments may be expressed by a single equation, as the following considerations will show. In Fig. 134 regard the lines A, B and C as axes, any two of which are at right-angles to each other. Along A, say, arrange a scale of mass, along B a scale of the radius of the path of the centre of

gravity of the rotating body, and along C a scale of [rotational speed]². O is the common zero of all the scales. Select the results of any one of the experiments recorded and set out the respective values of the mass, the radius, and the square of the rotational speed Regard the points thus obtained—which are emphasised by black dots in the figure—as the corners of a solid possessing six rectangular faces. Regard the volume of this solid as being proportional to the radial force recorded in the selected experiment.

The graphs (Figs. 130 to 133) and their equations tell us that if we now increase the dimension along any one axis, the radial force will increase proportionately. The dotted lines in the diagram

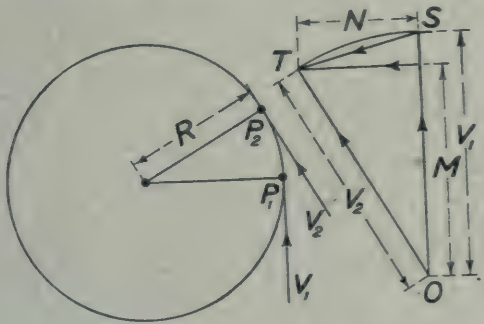

FIG. 135.

show that the volume of the solid also increases in the same proportion. If, however, we choose simultaneously to increase the dimensions along two of the axes or along all three, the increased volume which results must still be regarded as proportional to the radial force. The volume of a solid of the form in question is expressed by the product of the three dimensions along the axes and in consequence the radial force is proportional to the product of the mass, the radius, and the square of the rotational speed.

If a number A is caused to increase at the same rate as another number B, then if at any instant

$$\frac{A}{B} = K,$$

K being some constant number, the equation

$$A = KB$$

will hold good for every possible pair of values of A and B. Applying this to our experimental results, we have

$$\text{radial force} = K \times \text{mass} \times \text{radius} \times [\text{rotational speed}]^2.$$

The experiments have revealed what takes place under certain

conditions; an appeal to reason must be made to discover why these things occur. In Fig. 135 assume that all the rotating metal attached to one of the arms D (Fig. 129) is condensed into a point which moves with a constant tangential speed of V ft. per sec. around a circle of R ft. radius. Let the vector V_1 represent the velocity of the moving point when it occupies the position P_1, and the vector V_2 represent the velocity at the position P_2. The length of V_2 is equal to the length of V_1 because the tangential speed V does not alter; but none the less, V_2 represents a *velocity* which is not the same as V_1.

Resolve V_2 into vertical and horizontal components M and N respectively. It is then evident that V_2 possesses a horizontal component which V_1 does not, whilst its amount in the vertical direction is less than V_1. The total change from V_1 to V_2 might have been effected by adding a velocity, represented by the vector ST, to V_1, and had this been the case the rate at which ST was added would have been an acceleration imparted to the moving point.

The alteration is, however, effected in a somewhat different manner. As the moving point swings round from P_1 to P_2 the tangent to the circle, which represents instantaneously the direction of movement of the point, swings round at the same angular speed as the radial line through the moving point; so also may the vector which represents the velocity of the point be regarded as swinging round the centre O, and this it will do at the same angular speed as the tangential and radial lines.

It is usual and convenient to express the angular speed of an arm rotating about one of its ends in terms of the number of radians per sec. described by the arm, and to use the Greek letter ω (omega) to denote the angular speed. Adopting this symbol, we have both the radial line at the outer end of which the point P is situated and the vector V rotating at the rate of ω radians per sec. *The end of the vector describes a circle of radius V and the rate at which this circle is described is the rate at which the velocity represented by the vector changes.*

It is manifest that the change of velocity takes place at a constant rate and in consequence the moving point is subjected to a constant acceleration. The acceleration may be expressed in terms of the other quantities. If an arm of length V rotates at ω radians per sec. the linear speed of the outer end is ωV length units per sec. V representing ft. per sec., an acceleration is therefore imparted to the moving point at the rate of ωV ft. per sec. per second.

The tangent to the circle of radius V at the end of the moving vector is always parallel to the radial line drawn through the moving point. The acceleration is imparted at each instant in the direction of movement of the end of the vector, and as this coincides with the tangent referred to the acceleration is always toward the centre of the circular path of the point. As any radial line is

normal to the circular path the acceleration is accordingly termed a *normal acceleration.*

V ft. per sec. being the tangential speed of the point moving at the end of an arm of length R ft., which has a rotational speed of ω radians per sec.,

$$V = \omega R.$$

The normal acceleration was found to take place at the rate of ωV ft. per sec. per second. Substituting the alternative expression for V the normal acceleration may be expressed as $\omega^2 R$.

In the experiments recorded it was not a mere point, but a heavy body which was subjected to this normal acceleration and when a represents the acceleration produced in a body weighing W lb. by a force of F lb.,

$$F = \frac{W}{g} a.$$

Applying this to the present case, $\frac{W}{g}$ is the *mass* of the rotating body, $\omega^2 R$ gives the value of the acceleration a, and F, the radial pull indicated by the spring-balance shown in Fig. 129, is the force producing the acceleration. Substituting these values and restating the equation,

$$F = \frac{W}{g} \omega^2 R.$$

Alternatively we may express F in terms of V, the tangential speed, for since V equals ωR,

$$\omega = \frac{V}{R}, \text{ and } \omega^2 = \frac{V^2}{R^2}.$$

Substituting the last expression in the previous equation,

$$F = \frac{WV^2}{gR}.$$

The radial force F, measured by the apparatus, acting *toward* the axis of rotation of the heavy body and preventing it from leaving its circular path, is called a *centripetal force.* The inertia of the moving body, which is the resistance it offers to acceleration, acts *outward* from the centre of curvature, balances the centripetal force, and is known as *centrifugal force.* As centrifugal force and centripetal force are equal, the expression which represents one also represents the other.

That the equation found by reasoning is identical with the one obtained by experiment, viz.,

$$\text{radial force} = K \times \text{mass} \times \text{radius} \times [\text{rotational speed}]^2$$

is shown as follows :—

Let N represent the number of revs. per min., W the weight in lb. of the rotating body, R the radius of the path of the centre of gravity in ft., and F the centrifugal force in lb. The equation to the graph (Fig. 133) expressed by these symbols, is

$$F = 0{\cdot}000985\ N^2.$$

This result is obtained by using a body weighing 3·19 lb. rotating in a circle of 0·9 ft. radius. The equations to the graphs in Figs. 130 and 131 show that F is proportional to the weight W and the radius R. For a body weighing 1 *lb.* and rotating in a circle of 1 *ft.* radius, we have, therefore,

$$F = \frac{0{\cdot}000985}{3{\cdot}19 \times 0{\cdot}9} N^2 = 0{\cdot}000343\ N^2.$$

Now, let the other equation

$$F = \frac{W}{g}\omega^2 R$$

be modified in the same manner. As ω represents radians per sec.,

$$\omega = \frac{2\pi N}{60},$$

and

$$\omega^2 = \frac{\pi^2 N^2}{900}.$$

W and R are each to have unit value; substituting for ω^2 its value as given above,

$$F = \frac{1}{g} \times \frac{\pi^2 N^2}{900} \times 1 = 0{\cdot}00034 N^2.$$

This being almost exactly equal to the experimental result, sufficient evidence has been presented to establish the correctness of the reasoning. Finally, since F increases in direct proportion to the increase of the weight of the body (W lb.) and the radius of the path (R ft.); and as the last equation applies when W and R are each of unit value,

$$F = 0{\cdot}00034 WRN^2.$$

Circular Motion and S.H.M.—In the equation given on p. 183, viz.,

$$F = \frac{WV^2}{gR}$$

$\frac{V^2}{R}$ represents the normal acceleration of a point moving round a circle of radius R ft. in terms of the linear speed of the point. Compare this with the result given on p. 171. In the case to which reference is made, the movement of the reciprocating parts of a steam-pump was considered, and by making certain assumptions,

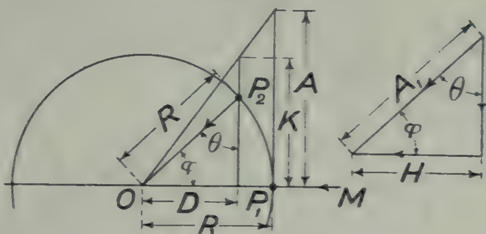

FIG. 136.

the conclusion was reached that if this movement was controlled by a crank-pin moving around a circle of radius R at a uniform speed V, the acceleration of the reciprocating parts would be represented by the expression $\frac{V^2}{R}$ at the commencement of the stroke.

In Fig. 136 let P_1 be the position of the crank-pin at the commencement of the stroke. The normal acceleration of the crank-pin is along MO and is shared by the reciprocating parts. At P_1 set up a vertical ordinate A to represent the normal acceleration $\frac{V^2}{R}$. Join the upper end of the ordinate to O. Let the crank-pin now move to another position, P_2. Through P_2 draw a vertical ordinate K. If the reciprocating parts really possess S.H.M., the acceleration toward O being proportional to the length A at the commencement of the stroke, then the length K represents their acceleration when the crank-pin is at P_2. The point in question is: Does the movement of the crank-pin at a uniform speed V actually produce S.H.M. in the reciprocating parts? At a previous stage we reached the conclusion that it does (p. 171), but only by making certain assumptions. We now know that at P_2 there is a normal acceleration along P_2O acting on the pin. Any horizontal effect which this acceleration may possess is shared by the reciprocating parts.

On the right of the figure a vector A_1 is drawn to represent the

normal acceleration $\frac{V^2}{R}$ at P_2. This is resolved into a vertical and a horizontal component, the latter being marked H. If H is equal to K then we may be quite certain that the assumptions previously made were well based and that S.H.M. actually does result from the circular motion. This is readily shown to be the case, for from the manifest equality of the angles marked θ and the equality of those marked ϕ.

$$\frac{H}{A_1} = \frac{D}{R} \text{ or } H = \frac{A_1 D}{R}.$$

Two similar triangles show that

$$\frac{K}{D} = \frac{A}{R} \text{ or } K = \frac{AD}{R}.$$

Since A and A_1 are equal, K and H are equal also, and the assumptions made prove to be justified.

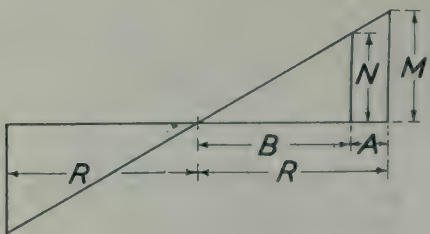

FIG. 137.

The reciprocating parts of the pump shown in Fig. 123 possess mass, and consequently any acceleration imparted to them implies the exertion of a force. The mass being constant, the force exerted at any instant is proportional to the acceleration imparted at that instant. After passing the mid-stroke point the acceleration changes from positive to negative (see Fig. 125). The accelerating force must also change from positive to negative, which means that instead of urging the moving parts along their path the accelerating force becomes a resistance. This resistance in the present instance is afforded by the pressure of the water in the pump-barrel.

The barrel of a steam-pump constructed as in Fig. 123 *has a diameter of* 4 *in. The diameter of the steam-cylinder is* 6 *in. The reciprocating parts weigh* 1 *cwt. The number of double strokes per min. is* 180. *The length of the stroke is* 10 *in. What is the pressure of the water in lb. per sq. in. when the piston has moved through one-tenth of its stroke, the steam pressure then being* 60 *lb. per sq. in.?*

The total force acting in the direction of movement is the pressure on the steam-piston. If P is this force,

$$P = \frac{\pi}{4} 6^2 \times 60 \text{ lb.} = 1{,}712 \text{ lb.}$$

P is balanced by two forces, one being the resistance due to water pressure, the other being the inertia resistance of the moving parts. Let V ft. per sec. be the tangential speed of the crank-pin and R ft. the length of the crank.

$$V = \frac{\pi 10 \times 180}{12 \times 60} = 7{\cdot}85 \text{ ft. per sec.}$$

$$R = \frac{10}{2 \times 12} = 0{\cdot}417 \text{ ft.}$$

$$\frac{V^2}{R} = \frac{(7{\cdot}85)^2}{0{\cdot}417} = 147{\cdot}5 \text{ ft. per sec. per second.}$$

When the piston has moved through one-tenth of its stroke it has completed one-fifth of the half-stroke. From Fig. 137 it is evident that if A represents the portion of the stroke completed—$0{\cdot}2R$ that is—then B represents $0{\cdot}8R$ and

$$\frac{N}{M} = \frac{0{\cdot}8R}{R}$$

Hence,

$$N = 0{\cdot}8M.$$

M represents the acceleration at the beginning of the stroke, N the acceleration at the point stated in the problem. Therefore

$$N = 0{\cdot}8\frac{V^2}{R} = 0{\cdot}8 \times 147{\cdot}5 = 118 \text{ ft. per sec. per second.}$$

Substituting numerical values in the equation,

$$F = \frac{W}{g}a,$$

$$F = \frac{112}{32{\cdot}2} \times 118 = 410 \text{ lb. (approx.)}$$

$$P - F = (1{,}712 - 410) \text{ lb.} = 1{,}302 \text{ lb.}$$

If the required water pressure is p lb. per sq. in.,

$$P - F = \frac{p\pi 4^2}{4} = 1{,}302 \text{ lb.}$$

Hence

$$p = \frac{1302}{4\pi} = 104 \text{ lb. per sq. in.}$$

The assumption has been made that no rotational acceleration is imparted to the fly-wheel. As a matter of fact, the fly-wheel would be of no use if this were really the case at the point in the stroke considered; for the sake of simplicity this acceleration has not been taken into account.

Resultant Centrifugal Force.—In the experimental investigation of centrifugal force, the mass of the rotating body was regarded as *concentrated* at its centre of gravity. It may in consequence be inferred that this is a correct assumption in the case of any problem involving the mass of a moving body. Such a conclusion would be a grave error, and therefore it is desirable to justify the assumption in the example quoted.

Consider a sphere to be rotating at a uniform speed in a horizontal

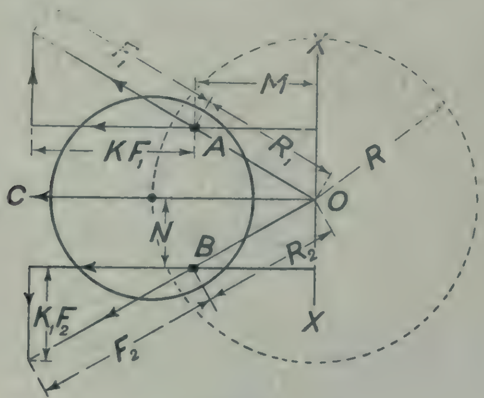

Fig. 138.

circular path about a vertical axis. In Fig. 138 O is a plan view of the axis of rotation. The path of the centre of the sphere is shown by a dotted circle. Imagine that the sphere is formed of an immense number of thin vertical wires of square section cemented together. The black dots A and B represent the horizontal sections of two such wires chosen at random. Every portion of one of these wires may reasonably be regarded as being at the same distance from the axis of rotation. The centrifugal force F_1 exerted by the mass of the wire A, in opposition to the inward radial pull necessary to keep it in its path, acts outward along the radius OA. Similarly, the centrifugal force F_2 of the wire B acts outwardly along the radius OB.

A similar effect is produced in the case of every other wire which helps to make up the sphere. These forces are not parallel, but as the sphere is a single body the resultant effect is a single force F acting outward along some radius OC. Assume that OC passes through the centre of gravity of the sphere.

Resolve F_1 along and at right-angles to OC. Let K be a fraction such that KF_1 represents the component of F_1 parallel to OC. Let M be the distance of A from a horizontal axis XX at right-angles to OC, and R_1 the distance of the wire A from the axis of rotation. Then by similar triangles,

$$\frac{KF_1}{F_1} = \frac{M}{R_1} \text{ or } K = \frac{M}{R_1}$$

If the weight of the wire A is W_1 lb. and if the rotational speed of the sphere is ω radians per sec.,

$$F_1 = \frac{W_1}{g}\omega^2 R_1,$$

and

$$KF_1 = \frac{M}{R_1} \times \frac{W_1}{g}\omega^2 R_1 = \frac{W_1}{g}\omega^2 M.$$

For each such wire as A a similar expression will be obtained. The resultant centrifugal force F of the complete sphere is equal to the sum of all such components as $\frac{W_1}{g}\omega^2 M$, the factor $\frac{\omega^2}{g}$ being constant. The variable factors are represented by W_1M, and if all the different values of W_1M are added together the result is equal to the sum of the moments about the axis XX of the weights of all the constituent parts of the sphere ; this sum is equal to WR when W is the weight of the whole sphere, and R is the distance of its centre of gravity from the axis XX. Hence,

$$\frac{\omega^2}{g}\text{ (sum of all such terms as } W_1M) = \frac{\omega^2}{g}WR = F.$$

The first expression gives the resultant centrifugal force as the sum of a number of components, the second one gives it in terms of the total weight of the body and the distance of the centre of gravity from the axis of rotation.

Further reasoning is required to show that the line of action of the resultant force F actually passes through the centre of gravity. Consider the wire B, situated at a distance R_2 from O. Let the centrifugal force F_2 due to the wire B be resolved as previously and let K_1F_2 be the component of F_2 parallel to the axis XX. Then, referring to the diagram,

$$\frac{K_1F_2}{F_2} = \frac{N}{R_2} \text{ and } K_1 = \frac{N}{R_2}.$$

Let W_2 be the weight of the wire B. Then

$$F_2 = \frac{W_2}{g}\omega^2 R_2.$$

By substituting the given value of K_1 on the right-hand side of the equation we have

$$K_1F_2 = \frac{W_2}{g}\omega^2 N = \frac{\omega^2}{g} \times W_2N.$$

The factor W_2N is the moment of the weight of the wire B about the axis OC. As this axis is assumed to pass through the centre of gravity of the sphere, the sum of the moments of the weights of all the constituent wires on one side of it is equal to the corresponding sum with respect to all the wires on the other side. Maintaining the assumption, the sum of all such expressions as $\frac{\omega^2}{g} W_2N$ with regard to wires on one side of the axis must therefore equal the sum of those on the other side ; that is, since the two sets of components act in opposite directions the resultant of *all* the components is nothing. The sphere is assumed to be whirling round its path at a constant speed, but if any resultant force is exerted upon it in the direction of the components in question, acceleration of the sphere along its path will result. As no such acceleration occurs when the sphere is moving at a uniform speed, no accelerating force can be acting in a direction tangential to the path of the centre of the sphere. For this reason the algebraic sum of the components at right-angles to the axis OC must be nothing, a condition which is secured, as has been shown, when this axis passes through the centre of gravity. Reasoning in the same way, it may be shown that OC cannot pass through the sphere either above or below the centre of gravity. *Hence the line of action of the resultant centrifugal force really does pass through the centre of gravity.*

It has already been shown that *the resultant centrifugal force may be found by assuming all the mass of the body to be situated at some point which has a distance from the axis of rotation equal to that of the centre of gravity.* Hence in calculating centrifugal force we may justly regard the mass of a body as being concentrated at its centre of gravity.

It is true that we have assumed the body to be spherical in shape, but this is done simply to ensure a definite mental image ; there is no part of the reasoning which will not apply with equal force to a body of any shape whatever.

EXAMPLES

133. The length of a simple pendulum is 14 ft. Find the period of a complete vibration.
134. A simple pendulum takes 0·36 sec. to make a complete vibration. Determine its length in ft.

135. The length of a simple pendulum is 16 ft. If this pendulum is deflected through an angle of 30° to the vertical and then released, what will be the maximum speed attained by the bob?

136. A point moves around a circle of 3 ft. radius, fixed in a vertical plane, at the rate of 10 revs. per sec. When the radial line joining the point to the centre of the circle is at an angle of 60° to the vertical diameter of the circle, what will be the acceleration of the moving point, in a horizontal direction?

137. A piece of metal weighing 12 lb. is attached to the end of a helical spring and allowed to oscillate in a vertical line. If the period of one complete vibration is 0·4 sec., how much is the spring stretched when the weight upon it remains at rest?

138. A piece of metal weighing 10·5 lb. is attached to the lower end of a helical spring of which the upper end is fixed. A pull of 20 lb. will stretch the spring 6 in. Depressing the load, releasing it and allowing it to oscillate, how long will it take to perform one complete oscillation?

139. In a steam-pump of the construction shown in Fig. 123 the crank length is 4 in. and the weight of the reciprocating parts is 72 lb. If the crank makes 120 revs. per min., what is the maximum accelerating force acting upon the moving parts?

140. A governor ball, rotating in a circle of 1 ft. diameter at a speed of 100 revs. per min., exerts a centrifugal force of 16 lb. If the same ball is caused to rotate in a circle of 9 in. diameter and to exert the same centrifugal force, at what speed must it rotate?

141. A ball weighing 14 lb. is whirled around a circle of 8 ft. diameter: the tangential velocity is 24 ft. per min.; what centripetal pull must be exerted upon the ball to keep it in its path?

142. A balance weight weighing 100 lb. is bolted to the web of a crank, its centre of gravity being 15 in. away from the axis of rotation. If the axes of the bolts are at right angles to the axis of rotation, what pull is induced in the bolts when the crank rotates at the rate of 210 revs. per min.?

143. A circular metal disk, weighing 0·25 lb. per cub. in., is 2 in. thick and 18 in. in diameter. A circular hole, its axis parallel to and 6 in. distant from that of the disk, is formed in the disk. The diameter of the hole is 5 in. If the disk is rotated about its geometrical axis at a speed of 300 revs. per min., what lateral force due to lack of balance is exerted upon the spindle to which the disk is fixed?

144. Let the disk described in the preceding question be mounted upon an axis which is half-way between the axes of the hole and the disk and parallel to each. When rotating at 300 revs. per min., what will be the lateral pressure upon the spindle?

145. Two parallel shafts with axes 2 in. apart are connected by an Oldham coupling (Fig. 99) and rotate each at a speed of 240 revs. per min. The weight of the circular plate E (Fig. 99) is 50 lb. Assuming that no energy is transmitted, what is the pressure of one of the feathers on the plate against the side of the groove at the moment when the two feathers exert equal pressures ?

CHAPTER VIII

Tension in a Rotating Wheel-Rim.—Problems involving the action of centrifugal force may generally be solved without difficulty if the position of the centre of gravity of the rotating body is known. In Fig. 139 assume that the circle represents the rim of a pulley of slight thickness. When the pulley is rotating, every portion of the rim is subjected to an inward acting centripetal force, and therefore each portion of the rim must be pulled toward the centre of the wheel by the adjoining parts of the rim. Take, for example, the portion marked CD. F represents the centrifugal force, and the forces PP represent tangential pulls exerted by the neighbouring parts of the rim upon the part CD. The three forces are in equilibrium. The effect of the force P is to set up a tangential tension at every section of the rim. If the speed of rotation is unduly increased, the centrifugal force becomes great enough to overcome the strength of the rim, and the wheel will then fly to pieces. Hence there is a speed limit for wheels beyond which they cannot be run with safety.

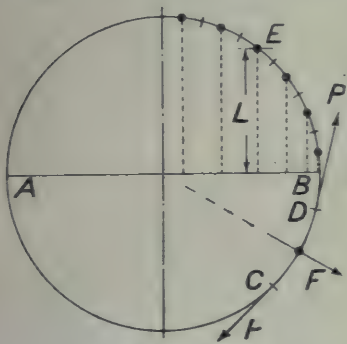

FIG. 139.

Let the rim be divided into two parts by any diameter AB. The resultant centrifugal force upon either half has a line of action which is at right-angles to the diameter AB; for it is evident that the centre of gravity of the half-rim is situated in a radial line which bisects it.

How far is the centre of gravity from the diameter AB? If the half-rim is divided into numerous parts of equal length, each part does not deviate materially from straightness. Neglect the deviation and assume that each part is absolutely straight.

In Fig. 139 one quarter of the circle is divided into six equal portions. Let w lb. be the weight of each. The centre of gravity of each part coincides with its middle point. Taking the plane of the circle, for the time being, as horizontal, the turning moment of the weight of the part E, say, about the diameter AB is equal to wL. If the moments of the weights of all six parts are found and added together their sum will equal the total weight of the six

parts multiplied by the distance of their common centre of gravity from the axis AB.

The centre of gravity of the adjoining quarter-rim is situated at the same distance from AB, as also, of course, is the centre of gravity of the half-rim. Drawing a circle of 10 in. diameter and performing these operations, the following numbers were obtained :—

Number	1	2	3	4	5	6
Dimension L (in.)	0·65	1·9	3·04	3·96	4·61	4·96
Moment about AB (lb.-in.)	$0{\cdot}65w$	$1{\cdot}9w$	$3{\cdot}04w$	$3{\cdot}96w$	$4{\cdot}61w$	$4{\cdot}96w$

The sum of the moments is $19{\cdot}12w$, and putting R to represent the

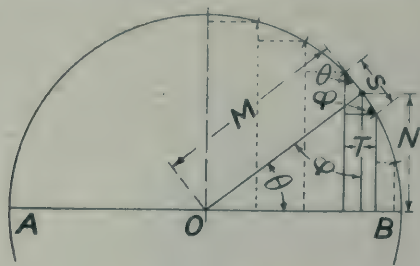

FIG. 140.

distance from AB of the centre of gravity of the half-rim,

$$19{\cdot}12w = 6w \times R.$$

Hence,

$$R = \frac{19{\cdot}12}{6} = 3{\cdot}186 \text{ in.}$$

Independently of any error of measurement, this value of R is merely approximate, for we have considered the portions of the rim as straight, which they are not.

A perfectly accurate method of finding R will now be described. The approximate method is given in order to inspire confidence in the results obtained by the second method. In Fig. 140 the quarter-rim is divided into six parts as before. Consider that part which is indicated by the dimension S as coinciding along the whole of its length with the tangent touching the circle at its centre. By simple geometrical reasoning it may be shown that the angles marked θ and ϕ are respectively equal and in consequence that

$$\frac{S}{T} = \frac{M}{N}.$$

Hence,

$$SN = MT.$$

All the dimensions in the last equation must be expressed in terms of some unit of length. Let K be the weight of a portion of the rim equal in length to the unit in question. Then KS is the weight of the portion of rim having the length S. The moment of this weight about AB is KSN, the dimension N being indicated in the figure. Since

$$SN = MT$$
$$KSN = KMT.$$

The equation applies to each of the six parts into which the quarter of the rim is divided. The sum of the moments about the diameter AB will therefore be the sum of all such quantities as KSN or KMT. (In expressing the sum of a number of terms of similar formation, the Greek letter Σ (sigma) is used, ΣKSN meaning "the sum of all such terms as KSN.") Let the remaining part of the half-rim above AB be divided up in the same way as the quarter of the rim already dealt with. Then the sum of the moments of the weight of all the parts of the half-rim may be expressed as ΣKSN or ΣKMT. (When factors are common to all the terms of an expression the result may be stated by placing the common factors to the left of the symbol Σ to indicate that the sum of the terms on the right is to be multiplied by the common factors on the left.) Hence,

$$K\Sigma SN = KM\Sigma T.$$

Now the length of the half-rim is ΣS; hence the weight of the half-rim is KΣS, and if R is the distance from AB of the centre of gravity of the half-rim, KRΣS is the moment about AB of the weight of the half-rim. Hence,

$$KR\Sigma S = K\Sigma SN = KM\Sigma T$$

or

$$R\Sigma S = M\Sigma T$$

Hence,

$$\frac{R}{M} = \frac{\Sigma T}{\Sigma S}.$$

Still considering the half-rim, if all such lengths as T are projected on to AB it is clear that ΣT is the diameter of the rim. ΣS is the semi-circumference of the rim. M being the radius of the rim and R the distance from AB of the centre of gravity of the half-rim, the last equation becomes, by substitution for the symbols employed,

$$\frac{\text{dist. from AB of centre of gravity of half-rim}}{\text{radius of the rim}} = \frac{\text{diameter}}{\text{semi-circum.}} = \frac{2}{\pi}.$$

Consequently the distance of the centre of gravity of the half-rim

from AB is $\frac{2}{\pi} \times$ radius of the rim. Apply this result to the example illustrated by Fig. 139. Here the radius of the circle is 5 in. The distance of the centre of gravity of the half-rim is therefore

$$\frac{2}{\pi} \times 5 \text{ in.} = \frac{10 \text{ in.}}{\pi} = 3{\cdot}121 \text{ in.}$$

The approximate method gave 3·186 in., which shows no great deviation from the result obtained by the exact method.

At first sight it may appear that the second method is no more accurate than the first one, but it must be taken into account that the final expression does not depend upon dividing the quarter circle into six parts. Had we taken sixty parts instead of six, the reasoning would apply equally well and the deviation from perfect straightness of each of the short lengths S would then have been much less than it is when their number is small. If, instead of sixty parts, the number taken is inconceivably large, all deviation from straightness of each part disappears and in consequence any error due to lack of straightness also disappears. Since the final result was obtained without limiting the number of parts considered, the result must be absolutely exact.

At what speed will the tension produced in the rim of a rotating pulley by centrifugal force be sufficient to overcome the resistance of the material?

Let R_1 ft. be the mean radius of the pulley rim, A sq. in. be the sectional area of the rim, K lb. be the weight per cub. in. and f the ultimate tensile stress of the material in lb. per sq. in. The number of cub. in. of metal in the half-rim is $12\pi R_1 A$; the factor 12 occurs because R_1 ft. must be converted to inches. Hence the weight of the half-rim is $12K\pi R_1 A$ lb. The distance from the axis of rotation of the centre of gravity of the half-rim is $\frac{2}{\pi} R_1$ ft. Let ω represent the rotational speed in radians per sec. Then if F lb. is the total centrifugal force due to the half-rim the general equation

$$F = \frac{W}{g} \omega^2 R,$$

in which R represents the radius of the centre of gravity, becomes, after substituting the values given,

$$F = \frac{12K\pi R_1 A}{g} \omega^2 \times \frac{2}{\pi} R_1,$$

or

$$F = \frac{24KR_1^2 A\omega^2}{g}$$

in which R_1 is the mean radius of the rim.

Considering all the forces acting upon the upper half of the rim (Fig. 141), we have the force F balanced by two equal pulls PP due to the tensile resistance of the material. Since $P = \frac{1}{2}F$,

$$P \text{ (lb.)} = \frac{12KR_1^2A\omega^2}{g} \text{ (lb.)}$$

P may be expressed in terms of the sectional area of the rim and the tensile stress. Consequently

$$fA = \frac{12KR_1^2A\omega^2}{g},$$

$$f = \frac{12KR_1^2\omega^2}{g}$$

and

$$\omega^2 = \frac{fg}{12KR_1^2}.$$

Expressing the rotational speed as N revolutions per min.,

$$\omega = \frac{2\pi N}{60}.$$

Hence,

$$\omega^2 = \frac{\pi^2N^2}{900} = \frac{fg}{12KR_1^2},$$

or

$$N^2 = \frac{900fg}{12\pi^2KR_1^2} = \frac{245\ f}{KR_1^2}.$$

The last equation shows that although an increase of the thickness of the rim would proportionately increase its strength, yet this will not permit us to increase the speed, because a proportionate increase of mass and consequently of centrifugal force also would follow. For a pulley of given radius, an allowable increase of speed is obtained either by using a stronger material, that is, by increasing f; or by using a less dense material, that is, by making K smaller. If f is taken as the greatest *allowable* stress instead of the ultimate, or breaking, stress, ω will represent the maximum speed at which the pulley may be run. For cast-iron, f may be taken as 1,000 lb. per sq. in., and K as 0·26 lb. per cub. in. Substituting in the equation

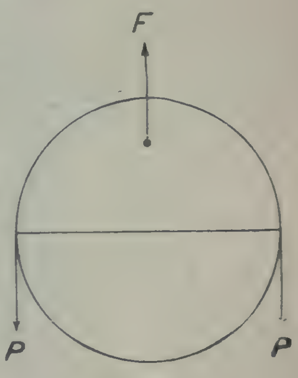

FIG. 141.

$$\omega^2 = \frac{fg}{12KR_1^2}$$

$$\omega^2 = \frac{1{,}000\,g}{12 \times 0{\cdot}26 \times R_1^2} = \frac{10{,}000}{R_1^2} \text{ (approx.)}$$

From this

$$\omega^2 R_1^2 = 10{,}000$$

$$\omega\ R_1 = \sqrt{10{,}000} = 100.$$

As ωR_1 represents the tangential speed in ft. per sec., we have 100

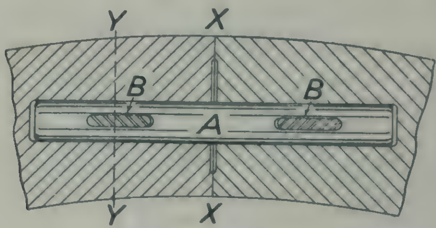

FIG. 142.

ft. per sec. as the greatest allowable tangential speed of a cast-iron pulley rim.

Two wheels are made from the same pattern, one of steel, the other of an aluminium alloy. The density of the aluminium alloy is one-third that of the steel, its safe tensile stress being 6 tons per sq. in., whilst that of the steel is 2 tons per sq. in. What is the ratio of the limiting speed of safety of the steel wheel to that of the other?

From the preceding example we have

$$\omega^2 = \frac{fg}{12KR_1^2}$$

when f is the safe stress, R_1 is the mean radius of the rim, and K is the weight per cub. in. of the material. Let ω_1, f_1 and K_1 apply to the steel wheel, and ω_2, f_2 and K_2 apply to the other wheel. The other factors of the equation are common to both wheels. The ratio of the respective densities of the two materials is $K_1 : K_2$. Hence,

$$\frac{\omega_1^2}{\omega_2^2} = \frac{f_1}{K_1} \div \frac{f_2}{K_2} = \frac{f_1 K_2}{f_2 K_1} = \frac{2 \times 1}{6 \times 3} = \frac{1}{9}.$$

$$\frac{\omega_1}{\omega_2} = \sqrt{\frac{1}{9}} = \frac{1}{3}.$$

Whether we consider rotational speed, or the linear speed of the rim, the wheel made of the aluminium alloy may therefore be run at three times the speed of the steel wheel.

For various reasons the rim of a very large fly-wheel is usually made in several parts ; any device adopted for joining these parts must provide for the tangential tension in the rim. In Fig. 142 is shown a type of joint frequently employed. A represents a pin of circular section, embedded in the rim. The plane of the joint between two parts of the rim is shown at XX. Two cotters BB are passed through slots in the rim and in the pin A. The cotters are usually riveted over at each end. In designing such a joint the force tending to shear the cotters and pull the pin in two must be determined.

The rim of a fly-wheel weighing 12 *tons and divided into eight sections has a mean radius of* 10 *ft. The speed of rotation is* 80 *revs. per min. If the sections of the rim are joined as shown in Fig.* 142, *what must be the net section of the pin A (taken in the section* **YY**, *Fig.* 142) *if the metal of which it is made will safely sustain a tensile stress of* 4 *tons per sq. in. ?*

Let R ft. be the distance of the centre of gravity of half the rim from the axis of rotation.

$$R = \frac{2}{\pi} \times 10 \text{ ft.} = 6{\cdot}375 \text{ ft.}$$

If W is the weight of the half-rim

$$W = \frac{12 \times 2{,}240}{2} = 13{,}440 \text{ lb.}$$

If F lb. is the centrifugal force due to the half-rim

$$F = \frac{W}{g}\omega^2 R = \frac{13{,}440 \times (8{\cdot}37)^2 \times 6{\cdot}375}{32{\cdot}2} = 187{,}000 \text{ lb.}$$

The force F is divided between two pins, one at each end of a diameter of the wheel, so that the force on each pin is $\frac{1}{2} \times 187{,}000$ lb. or 93,500 lb.

$$\text{Area of section of pin} = \frac{\text{force acting on the section}}{\text{safe tensile stress}} \cdot$$

$$= \frac{93{,}500}{4 \times 2{,}240} = 10{\cdot}4 \text{ sq. in.}$$

The number of parts into which the rim is divided will in no way affect the size of the pin.

Tension in Driving Belts due to Centrifugal Force.—Let the circle in Fig. 143 represent a rotating pulley connected by a driving belt to a larger pulley, not included in the figure. Disregarding the thickness of the belt, which is never considerable, all the curved

part moves around a circular path of M ft. radius. Since the belt possesses mass, a centripetal force is required to cause it to travel along the circular path, and hence there will be a resultant centrifugal force F lb. in the direction indicated.

Equal pulls of P lb. each along the two straight parts of the belt will be necessary in order to balance the force F. The three forces are represented in the figure by a vector triangle. Let us now find an expression which gives the amount of F. Referring to p. 195 we find that if one-half of a heavy ring is considered, the ratio of the distance of its centre of gravity from the centre of the ring with respect to the radius of the ring is equal to the ratio of the diameter to the semi-circumference. The latter ratio was first found in the

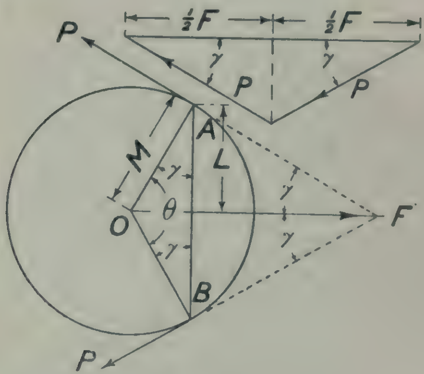

FIG. 143.

form $\Sigma T : \Sigma S$, when T was obtained by projecting a short length S of the circle on to the diameter.

A brief examination of the reasoning employed will reveal the fact that a similar result is obtained if the arc AB to the right of the chord AB in Fig. 143 is projected upon the chord AB. Let R be the distance from O of the centre of gravity of the curved part of the belt and θ radians be the angle subtended by the curved part of the belt, M being the radius of curvature and the length of half the chord being represented by L.

$$\frac{R}{M} = \frac{\text{length of chord AB}}{\text{length of arc AB}} = \frac{2L}{\theta M}.$$

Hence,

$$R = \frac{2L}{\theta}.$$

Let K lb. be the weight of 1 ft. length of the belt. Then the weight

W of the curved part is $K\theta M$. Let ω radians per sec. be the rotational speed of the pulley. Substituting the different values in the general equation, viz.,

$$F = \frac{W}{g}\,\omega^2 R,$$

we have

$$F = \frac{K\theta M}{g}\,\omega^2 \times \frac{2L}{\theta} = \frac{2KML\omega^2}{g}.$$

Consider the relationship between F and the tensile force P (Fig. 143). Examination of the figure will show that all the angles marked by the Greek letter γ (gamma) are equal. In consequence,

$$\frac{P}{\frac{1}{2}F} = \frac{M}{L}$$

and

$$P = \frac{FM}{2L}.$$

Substituting the value obtained for F,

$$P = \frac{2KML\omega^2}{g} \times \frac{M}{2L} = \frac{KM^2\omega^2}{g}.$$

The angle θ does not appear in the final expression, and hence the pull due to centrifugal force does not depend upon the arc of contact of the belt.

Now consider the other pulley, which is not shown in the figure. Let its radius be M_1 and its rotational speed be ω_1. If V ft. per sec. be the linear speed of the belt

$$V = \omega M = \omega_1 M_1.$$

Consequently

$$\omega^2 M^2 = \omega_1{}^2 M_1{}^2$$

and

$$P = \frac{K}{g}\,M^2\omega^2 = \frac{K}{g}\,M_1{}^2\omega_1{}^2.$$

The last point is brought forward in order to show that it does not matter which of the two pulleys is chosen.

The capacity of a driving belt to transmit energy is materially influenced by the existence of tension due to centrifugal force. If a belt did not possess mass, the energy transmitted would be in direct proportion to its speed. As no belt can be without mass, the centrifugal tension, increasing in the same proportion as the square

of the rotational speed, must be deducted from the strength of the belt. Of the remainder, a certain proportion is required to balance the pull exerted along the slack part of the belt which is necessary to secure a frictional grip on the pulley.

These relationships are exhibited graphically in Fig. 144. At any point on the horizontal scale, the ordinate K represents the number of pounds weight which the belt is capable of sustaining, that is, the strength of the belt. As the strength K does not depend upon the speed, the graph M is a horizontal line. The graph O represents the tension due to centrifugal force; consequently, F is the force to be deducted from K in order to obtain the force T acting along the tight part of the belt and which is partially balanced by the pull S in the slack side of the belt. The experiment recorded on p. 124 shows that when slipping of the belt is at the point of taking place, the ratio of T to S is constant for different values of T and S. At any point in the horizontal scale therefore, a vertical intercept, such as E, between the graphs M and N represents the effective tension in the belt.

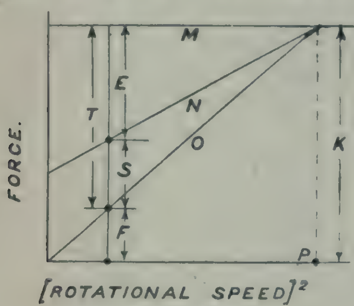

FIG. 144.

The point P on the horizontal scale indicates the square of the speed at which E diminishes to nothing. No energy can be transmitted when this speed is attained, for the belt strength is then fully utilised in opposing the tension produced by centrifugal force. E has its greatest value when the speed is nothing, and this again is a point at which no energy is transmitted. At some intermediate point the rate of energy transmission has a maximum value beyond which any increase due to running the belt at a greater speed is more than counterbalanced by the effect of centrifugal force. This fact may be illustrated by an example.

A leather belt weighing 0·6 *lb. per ft. is driven by a pulley of* 4 *ft. diameter. The greatest tension which the belt is strong enough to sustain is* 400 *lb. After deducting the force required to counterbalance the tension induced by centrifugal force,* 0·65 *of the remainder of the belt strength is equal to the effective tension. At what speed does the effective tension disappear and what is the greatest H.P. the belt will transmit?*

These data will apply to a single leather belt about 6 in. wide, embracing rather less than half the pulley. Let ω be the speed in radians per sec. at which the tension due to centrifugal force is sufficient entirely to balance the strength of the belt. If P be the force in question, it is equal to K in Fig. 144, or

$$P = K = 400 \text{ lb.}$$

From the equation on p. 201 in which P is expressed in terms of the speed and the weight per ft. of the belt,

$$P = 400 = \frac{0{\cdot}6 \times \omega^2 \times 4}{g}$$

$$\omega^2 = \frac{32{\cdot}2 \times 400}{0{\cdot}6 \times 4} = 5{,}366$$

$$\omega = \sqrt{5366} = 73{\cdot}2 \text{ radians per sec.}$$

If the speed of the belt is V ft. per sec. and R ft. is the radius of the pulley,

$$V = \omega R = 73{\cdot}2 \times 2 = 146{\cdot}4 \text{ ft. per sec.}$$

This is the required result which corresponds to a speed of 8,784 ft. per min.

In order to find the speed at which the greatest H.P. is transmitted, calculate and plot values of the H.P. corresponding to different speeds. At any speed of ω radians per sec.,

$$P = \frac{0{\cdot}6 \times \omega^2 \times 4}{g} = 0{\cdot}0746\, \omega^2.$$

Referring to Fig. 144 we see that if at any speed

$$E = mT, \quad m = \frac{E}{T}.$$

The stated value of m is 0·65, and as the ordinate K is 400 lb.,

$$\begin{aligned} E &= 0{\cdot}65\,(400 - P) \\ &= 0{\cdot}65\,(400 - 0{\cdot}0746\omega^2) \\ &= 260 - 0{\cdot}0485\omega^2. \end{aligned}$$

If V is the belt speed in ft. per sec.,

$$\text{H.P.} = \frac{E \times V}{550} = \frac{E \times \omega R}{550}.$$

Since $R = 2$ ft.,

$$\frac{\omega R}{550} = \frac{2}{550}\,\omega = 0{\cdot}00364\omega.$$

Hence,

$$\begin{aligned} \text{H.P.} &= E \times 0{\cdot}00364\omega \\ &\ (260 - 0{\cdot}0485\omega^2)\, 0{\cdot}00364\omega \\ &\ 0{\cdot}946\omega - 0{\cdot}0001765\omega^3. \end{aligned}$$

From the final equation, the following values of the H.P. have been calculated; these are plotted in Fig. 145.

ω (radians per sec.)	10	20	30	40	50	60	70
H.P. . . .	9·28	17·51	23·62	26·54	25·2	18·76	5·0

The graph (Fig. 145) shows that the maximum H.P. is approxi-

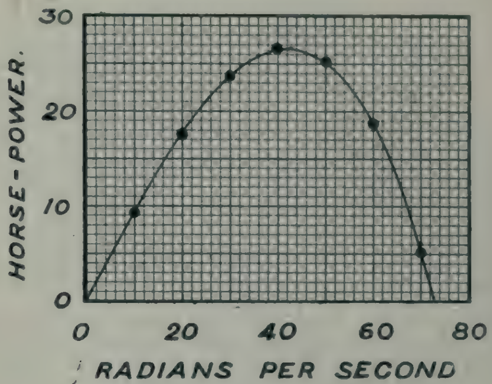

FIG. 145.

mately 26·6 and that this occurs when the speed is 42 radians per sec. or 400 revs. per min. The belt speed will then be 84 ft. per sec. or 5,040 ft. per min.

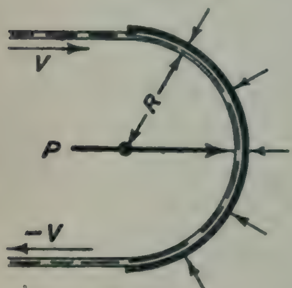

FIG. 146.

Impulse Wheels.—It will conduce to clearness of conception if the action of a jet of water upon a curved vane be dealt with by the method applied to a driving belt. In Fig. 146 a jet is shown impinging upon a fixed semi-cylindrical plate. Let the weight of each ft. of length of the jet be K lb. As the jet passes over the curved surface of the plate, each portion of it receives a normal acceleration. The inertia of the water therefore results in a normal reaction—centripetal force—being exerted by the plate. Let P be the resultant pressure of the water upon the plate. P is found in the same way as when a moving belt was in question, for this pressure is due to the resultant centrifugal force of the water in contact with the plate; con-

sequently it is expressed by the equation, given in connection with a moving belt weighing K lb. per ft. of length, on p. 201, viz.,

$$F = \frac{2KML\omega^2}{g}$$

In Fig. 146, R ft. denotes the radius of curvature of the plate; in Fig. 143 the angle of contact of the belt is marked θ. The angle of contact—equivalent to θ in Fig. 143—of the curved stream of water touching the plate in Fig. 146 is 180°; in the last equation, therefore, R now takes the place of both M, the radius of the pulley, and L, the length of half the chord of the arc of contact, which are shown in Fig. 143. F in the last equation is replaced by P in the present instance. Substituting R^2 for ML and P for F, the following equation expresses the resultant pressure upon the plate—

$$P = \frac{2K\omega^2R^2}{g}$$

If the speed of the jet is V ft. per sec. and the water delivered is W lb. per sec.,

$$W = KV \text{ or } K = \frac{W}{V}.$$

Since,

$$\omega R = V$$
$$\omega^2R^2 = V^2.$$

The value of P may now be restated in terms of these alternative expressions, thus :—

$$P = \frac{2K\omega^2R^2}{g} = \frac{2WV^2}{Vg} = \frac{2W}{g}V.$$

The final expression agrees with the result to be found on p. 74, and expresses the fact that the pressure exerted upon the plate is equal to the change of momentum taking place in one sec. The change is found by subtracting the final momentum from the initial momentum. If the plate consists of less than half a cylinder, $\frac{W}{g}V_1$ in Fig. 147 represents the initial momentum per sec. and $\frac{W}{g}V_2$ represents the final momentum per sec. of the jet. Each quantity of momentum is proportional to the velocity of the jet, for the mass of water per sec. does not vary. As no change of speed occurs either, the lengths of the two vectors are equal; the change of momentum is the result simply of the change of direction. The subtraction of the final momentum from the initial momentum is performed by reversing the arrow of the vector representing $\frac{W}{g}V_2$ and adding the reversed vector to the one representing $\frac{W}{g}V_1$. Thus, in Fig. 147, P is the resultant pressure on the plate due ot

the change of momentum taking place in one sec. This fact is not argued out in detail, but may be apprehended by considering the analogous case illustrated by Fig. 143.

Substitute for the plate of circular curvature one which has some other continuous curve, but which the same jet may enter

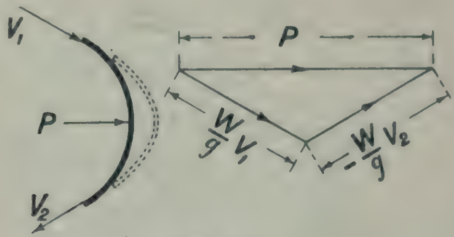

FIG. 147.

and leave tangentially along the same lines as before ; such a plate is represented by the dotted curve in Fig. 147. The vector figure which was drawn for the first plate applies to the second one also, and thus the resultant pressure on the second plate does not differ from that on the first one. Hence we are not concerned with the shape of the curve. The only difference between the action of the two plates is that the circular one imparts normal acceleration at a

FIG. 148.

uniform rate whilst the other imparts it at a different or, possibly, a non-uniform rate. The impulse imparted to the water by each of the plates is the same because the resulting change of momentum is the same.

To the curved plate of Fig. 146 let a speed of V_3 ft. per sec. be imparted as indicated in Fig. 148. The speed of entry being V_1 ft. per sec., the jet will slide around the plate at the rate of $V_1 - V_3$ ft. per sec., for when it has once touched the plate, normal acceleration is the only action to which it is subjected—if friction be excepted—and ***normal acceleration cannot exert any influence in a tangential direction.*** $V_1 - V_3$ is the speed of the jet relative to the

plate. Let V_2 be the final absolute speed of the water as it leaves the plate. Then V_2 is the algebraical sum of the speed of the plate and the speed of the water relative to the plate. The first of these quantities is V_3: the second is $-(V_1 - V_3)$ because the relative speed at leaving is in the backward direction. Hence,

$$V_2 = V_3 - (V_1 - V_3) = 2V_3 - V_1.$$

Let V_2 be nothing, then

$$0 = 2V_3 - V_1$$

and

$$V_1 = 2V_3.$$

This result justifies the following statement :—

If a jet impinges tangentially upon a curved plate moving forward

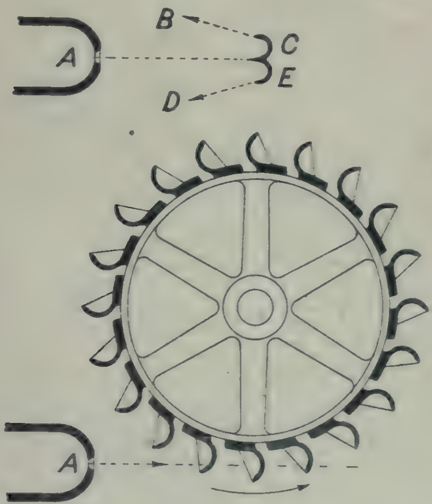

FIG. 149.

in the same direction as the jet at a speed equal to half that of the jet; and if the jet is so deflected by the plate that it leaves in an opposite direction along a line parallel to the one along which it entered, then the final absolute speed of the jet is zero.

As a consequence the final momentum of the jet is zero also. Moreover, the final K.E. of the jet is zero, so that all the energy it initially possessed has been imparted to the moving plate.

Fig. 149 represents a *Pelton water-wheel.* A jet of water issues horizontally from a nozzle A and impinges upon a sharp edge common to two curved "buckets." In the horizontal section at the upper part of the figure the dotted lines BC and DE show the

direction in which the divided jet leaves the buckets. The discharged water has therefore a velocity in the direction of the axis of rotation and consequently all the initial energy of the water has not been utilised. The residual energy enables the discharged water to clear the wheel. The possession of residual energy would alone be sufficient to prevent motors of this type realising the conditions dealt with in Fig. 148, and consequently working with an efficiency of 100 per cent.; but there are additional reasons why this is not the case. A material amount of friction occurs between the jet

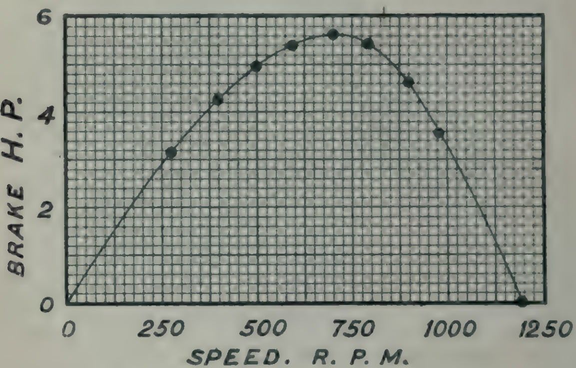

FIG. 150.

and the surface of the bucket. For this reason the most efficient speed of the buckets is rather less than half that of the jet. Furthermore, it is impossible to arrange that the jet shall continuously slide around the buckets in a perfectly tangential direction, and waste of energy will thereby result. To these losses the waste of energy due to mechanical friction and air resistance must be added. Consequently the efficiency of an actual Pelton wheel will not as a rule exceed 85 per cent. Experimenting with a Pelton wheel of which the supply nozzle had a diameter of $\frac{3}{4}$ in. with a head of water of 220 ft., the following observations were made :—

Revs. per min.	275	400	500	590	700	790	900	980	1190
B.H.P.	3·15	4·27	4·95	5·40	5·61	5·42	4·61	3·73	0

These numbers are plotted in Fig. 150. The graph indicates that the horse-power increases with the speed of rotation up to a certain point and then falls away to nothing. The effect is due to energy being rejected in the tailrace when the speed is either above or below that of the greatest efficiency.

Governors.—The operation of a prime mover, such as a turbine or a reciprocating engine, is due to a supply of energy. The stream

of energy that issues from the prime mover and is employed in performing work is the same one that entered, allowing for the energy consumed in overcoming frictional resistance. Of a number of machines driven by the prime mover, some may at certain times be more heavily loaded than at others; at one instant most of these machines may be at rest; at another they may all be at work. In consequence the rate of supply of energy must be continuously adjusted in accordance with the fluctuating rate of consumption. For reciprocating engines and turbines this result is achieved by using an appliance termed a "governor."

A governor usually depends upon the action of centrifugal force

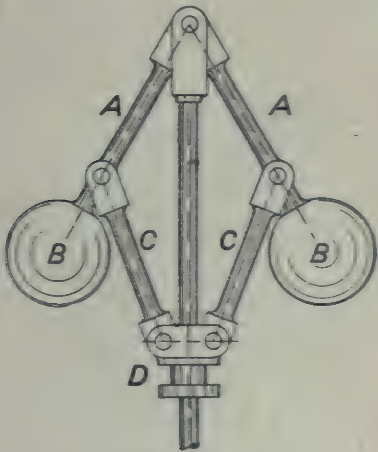

FIG. 151.

for the performance of its function. In the case of a steam-engine this function consists in varying the opening of a balanced valve, termed a throttle-valve, so as to alter the rate of flow of steam along the supply pipe, or in actuating mechanism which causes the sudden closing of a valve.

Fig. 151 represents a "*Watt*" *or pendulum governor.* Two arms AA are hinged at their upper ends to a vertical spindle. At the lower end of each arm a heavy ball B is fixed. Two links CC connect the arms to a "sleeve" D which is capable of sliding along the vertical spindle. The sleeve is connected to a lever which actuates the throttle-valve. If the balls are forced away from the spindle, the sleeve moves upward and turns the lever. When the governor is in action the spindle is caused to rotate by connecting it to the main shaft of the engine. The balls are carried around with the spindle, and as each one moves along a circular path it is subjected to the action of centrifugal force. In this position the balls are said to be "floating," and whatever forces act upon them are balanced.

If the rotational speed is increased the balls rise to a higher position, the sleeve being raised in consequence.

In Fig. 152 the vector figure represents the forces acting upon the ball. These forces are: F the centrifugal force resulting from the inertia of the ball, W the weight of the ball, and P the pull produced along the arm. The centre line of the moving arm generates a cone, termed the *cone of revolution*, of which the base coincides with the horizontal plane containing the path of the centres of gravity of the balls. That the height of the cone depends solely upon the rotational speed is readily shown.

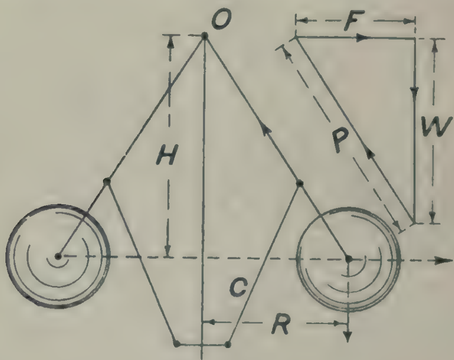

FIG. 152.

The forces F, P, and W (Fig. 152) being in equilibrium, their moments about O are balanced. The force P passes through O and therefore has no moment about that point. Hence the moments about O of F and W have equal magnitudes; that is,

$$FH = WR.$$

R is the radius of the centre of gravity of the governor ball, and H is the height of the cone of revolution. From the foregoing equation,

$$H \text{ (ft.)} = \frac{WR}{F} = WR \times \frac{g}{W\omega^2 R} = \frac{g}{\omega^2}.$$

The height H therefore varies inversely as the square of the speed of rotation, and if governors having balls of different weights and arms of different lengths are caused to rotate at the same number of revolutions per min., the arms will be inclined to the vertical at different angles, but the cones of revolution will all be of the same height.

No change of position of the ball can occur without a change of one of the forces acting upon it. Only an alteration of the centrifugal force can produce this effect, which in turn can only be caused

by a change in the rotational speed. The governor therefore possesses stability, which is a necessary quality, for otherwise there would be no means of controlling the throttle-valve so as to regulate the amount of steam flowing past it. Too high a degree of stability is not desirable, for the speed of the balls in their highest position—and in consequence the speed of the engine also—would then be considerably greater than in their lowest position. For many purposes, the speed variation of an engine must be kept within narrow limits.

In examining the action of a governor, the true criterion is the effect upon the sleeve resulting from any change of speed of the ball. Taking the governor in Fig. 152 as drawn to scale, the accompanying

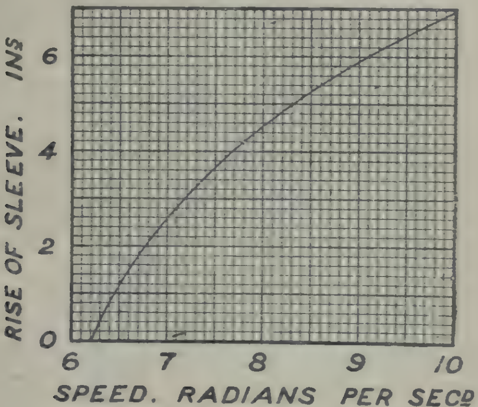

FIG. 153.

vector figure was constructed to correspond with a number of different positions of the sleeve. From the value found for F in each case, the rotational speed was calculated. The rise of the sleeve being plotted against the rotational speed, the graph (Fig. 153) was obtained. The slope of this graph is greater at the lower positions of the sleeve than at those which are higher; this indicates that for the higher positions of the balls a greater change of speed is required to cause a given movement of the sleeve than for the lower positions. As it is advantageous to cause as much movement of the sleeve as possible with the least amount of change of speed, it is evident that the working range of the governor should be confined to the lower positions of the ball.

Another matter is, however, equally important. Fig. 153 applies only when no force is exerted along the link C (Fig. 151). In reality some pull must be exerted along the link in order to overcome the resistance to movement of the throttle-valve and the mechanism connecting it to the governor. Suppose that whilst the balls are

floating, the three forces acting on each, viz., weight, centrifugal force and pull along the arm being in equilibrium, a resistance to movement of the sleeve is set up. In order to overcome this resistance an increase of speed must take place before any further rise of the sleeve occurs. Three more forces are now brought into

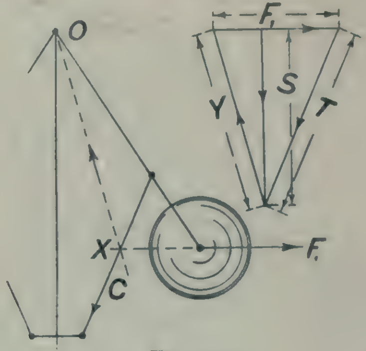

FIG. 154.

play, all of which act upon the governor *arm* and are in equilibrium. The forces in question are represented by the vector figure in Fig. 154. F_1 is the *additional* centrifugal force due to the increase of speed, T is a pull caused along the link connecting the arm to the sleeve, and Y is an additional reaction of the pin at O. The direction

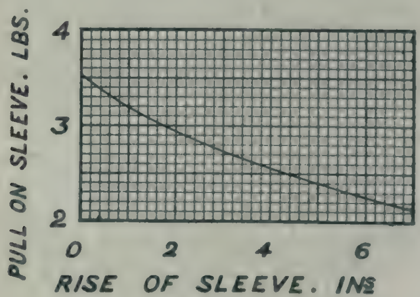

FIG. 155.

of the last force passes necessarily through the intersection of the lines of action of the first two.

For a given change of speed the additional centrifugal force F_1 at any stated position of the balls may be calculated and then, from the vector triangle, S the vertical component of the force T may be obtained. S is the pull on the sleeve, due to one of the balls, tending to overcome its resistance to movement. Assuming the two balls of the governor in Fig. 151 to possess a total weight of 10 lb. and that an increase of one radian per sec. occurred beyond the speed

required to keep the balls floating, the values of S for a number of different positions were found. The graph (Fig. 155) was obtained by plotting the values of S against the positions of the sleeve. Here we see that for a given change of speed the pull on the sleeve is greater for low positions than for high ones. Since it is desirable to exert as much force on the sleeve for as little change of speed as possible, we have therefore an additional reason for confining the working range of the sleeve to its lower positions.

Let two governor balls, A and B respectively, each weigh W lb. and rotate at the same speed of ω radians per sec. The path of A

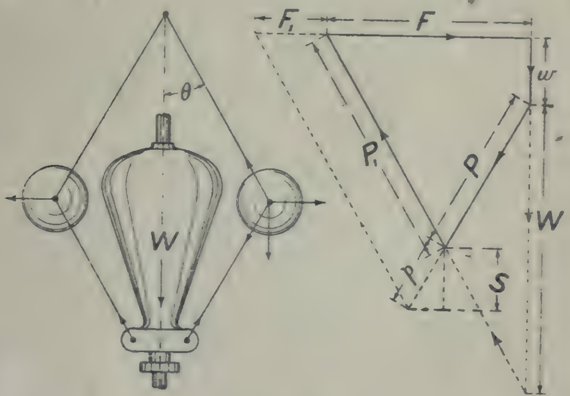

FIG. 156.

has a radius of R ft., that of B a radius of 2R ft. To each ball let an additional speed of 1 radian per sec. be imparted. Then for each ball,

additional centrifugal force = Final centrifugal force—Initial centrifugal force.

From this,

$$\frac{\text{centrifugal force added to A}}{\text{centrifugal force added to B}} = \frac{\frac{W}{g}(\omega + 1)^2R - \frac{W}{g}\omega^2R}{\frac{W}{g}(\omega + 1)^2 2R - \frac{W}{g}\omega^2 2R} = \frac{1}{2}.$$

A given increase of rotational speed causes therefore a much larger increase of centrifugal force in the case of the ball which is moving in the path of greater radius. One might expect to find in consequence that the pull on the sleeve also is greater for the higher position corresponding to the greater radius. That this is not so is due to the toggle action of the arm and the link.

Fig. 156 represents a "Porter" governor. In this type of

governor the high speed of the balls which is necessary to induce a considerable amount of centrifugal force is combined with the smaller values of the angle θ of the arms which are conducive to favourable action on the sleeve. The result is secured by loading the sleeve with a heavy piece of metal W, free to slide along the vertical spindle. The action of this weight is divided between the two balls. Assume the balls to be floating in the position given, and that all frictional resistance is absent. The *sleeve* will then be acted upon by three forces: an upward pull along each of the two lower links and the weight of the load.

By means of a vector triangle, two sides of which are dotted, the pull P along each of these links is found. The link being in tension, P is the amount of force which it exerts upon the ball. Four forces act upon each ball; w its weight, F the centrifugal force, a downward pull P along the lower link, and a pull P_1 along

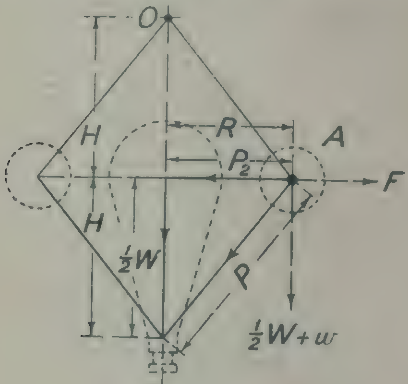

FIG. 157.

the upper link. The line of action of each force being given and the magnitudes of P and w being known, a vector figure may be drawn which shows the amount of F, the centrifugal force required to keep each ball floating. Knowing F, w, and the radius of the path of the centre of gravity of the ball, the speed of rotation may be found.

Assume now that a resistance of the sleeve comes into action and a change of speed occurs sufficient to induce *additional* centrifugal force amounting to F_1 lb. for each ball. Increase the vector representing F by the amount F_1 as shown in Fig. 156. The pull P is correspondingly increased by the amount p; S, the vertical component of p, is the upward pull on the sleeve due to one ball, which results from the change of speed. The total upward pull is, of course, 2S, since there are two balls.

The expression which gives the height of the cone of revolution for a pendulum governor does not apply to a Porter governor. Let

H ft. (Fig. 157) be the height in question, R ft. the radius of the centre of gravity of the ball, W lb. the weight of the central load, P lb. the pull along one of the lower links induced by the action of the load, F lb. the centrifugal force due to one ball, and w lb. the weight of one ball. If the length of the link be regarded as a vector expressing P, then H is the length of the vector expressing the vertical component of P, and R is the length of the vector expressing P_2, the horizontal component of P. Both components act through the centre of the ball A; $\frac{1}{2}$W lb. is the amount of the vertical component. Hence,

$$\frac{P_2}{\frac{1}{2}W} = \frac{R}{H} \text{ and } \therefore P_2 = \frac{WR}{2H}.$$

Of the forces acting upon the ball, the pull along the upper link

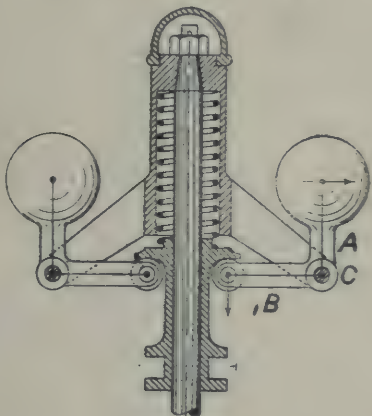

FIG. 158.

has no turning moment about the point O. Equating, with respect to O, the moments of the remaining forces,

$$(\tfrac{1}{2}W + w)R = (F - P_2)H.$$

Substituting for F and P_2

$$(\tfrac{1}{2}W + w)R = \left(\frac{w}{g}\,\omega^2 R - \frac{WR}{2H}\right)H.$$

From this we get

$$H\text{ (ft.)} = \frac{W + w}{w} \times \frac{g}{\omega_2}.$$

Fig. 158 represents a governor in which instead of a weight a spring is used for loading the sleeve. A bell-crank lever, of which

the arms are marked A and B respectively, turns about a pin at C fixed in a bracket which rotates with the vertical spindle. The arm B carries a small roller which bears upon the lower face of a collar projecting from the sleeve. The upper end of a helical spring, which is concentric with the spindle, bears against a collar at the top of the spindle, the lower end of the spring resting upon the sleeve. The movement of the arm A is restricted to a small angle on either side of the vertical. The length of the arm A is measured between the centre of the pivot C and the centre of gravity of the ball. The upward force on the sleeve due to both balls is counterbalanced by the downward pressure of the compressed spring.

Referring to Fig. 158, assume that each ball weighs 0·5 lb., and that when the arms occupy the position shown the rate of

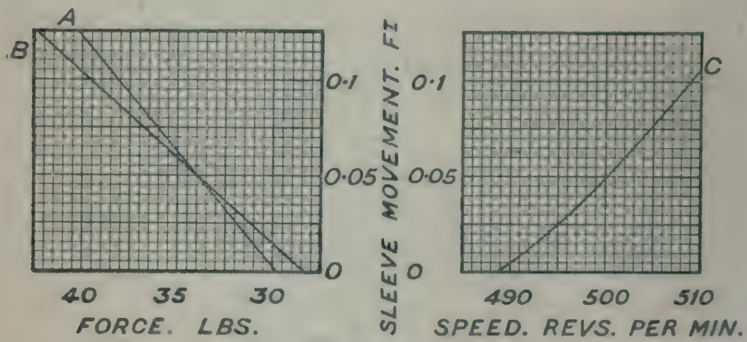

FIG. 159.

rotation of the governor is 500 revs. per min., the path of the centre of each ball then being a circle of 0·4 ft. radius. The arms of each bell-crank lever being of equal length, the downward force exerted by the spring is equal to the centrifugal force due to the two balls.

In accordance with these particulars,

$$\text{force exerted by the spring} = \frac{W}{g}\omega^2 R = \frac{1}{32\cdot 2} \times \left(\frac{500\pi}{30}\right)^2 \times 0\cdot 4 = 34 \text{lb.}$$

The experiment illustrated by Fig. 131 shows that when no change of rotational speed takes place the centrifugal force is proportional to the distance of the ball from the axis of rotation. Taking 0·35 ft. as the radius of the smallest ball-path and 0·45 ft. as the radius of the largest, we have, if no change of speed occurs,

$$\text{centrifugal force at inner position} = \frac{0\cdot 35}{0\cdot 4} \times 34 \text{ lb.} = 29\cdot 8 \text{ lb.}$$

$$\text{centrifugal force at outer position} = \frac{0\cdot 45}{0\cdot 4} \times 34 \text{ lb.} = 38\cdot 2 \text{ lb.}$$

The sleeve being in its zero position when each ball is 0·35 ft. from the axis of rotation, the numbers obtained afford data for constructing the graph A in Fig. 159, for any movement of the sleeve is due to an equal alteration of the ball-path radius.

If the helical spring in Fig. 158 is so compressed as to exert a force of 34 lb. when the balls are as shown, and if the graph A (Fig. 159) coincides with the load-extension graph of the spring, then, without any change of rotational speed, the force exerted by the spring will exactly balance the centrifugal force in whatever position the balls may be. This is due to the equality of the arms of each bell-crank lever, but if these were unequal the same effect might be obtained by modifying the compression and the rate of extension of the spring. A governor constructed in accordance with these suppositions would possess no stability, for the position of the sleeve would be unrelated to the amount of load on the engine. If a governor had no stability at all the balls would fly to their extreme outward position at the least impulse; the throttle-valve would then close, the speed would diminish, and the balls, returning to their inward position, would cause the throttle-valve to open and allow the speed to increase once more. This action is termed "hunting." Instead of maintaining uniformity of speed, an unstable governor would therefore induce an exactly opposite effect. In order to prevent "hunting," the spring is always made stiffer than we have supposed.

In Fig. 159 let the graph B represent the relationship between force applied to and the resulting compression of the spring shown in Fig. 158. When the radius of the ball-path is 0·45 ft., the sleeve is 0·1 ft. above its lowest position and the spring will then exert a force of 39·6 lb. At a speed of 500 revs. per min. the centrifugal force has been found to be 38·2 lb. for the stated position of the ball, and hence the speed must be increased in order that sufficient centrifugal force may be exerted to compress the spring. If N revs. per min. is the speed required,

$$\frac{1}{32 \cdot 2} \times \left(\frac{\pi N}{30}\right)^2 \times 0 \cdot 45 = 39 \cdot 6 \text{ lb.}$$

From the above we find N to be 509 revs. per min. The pressure of the spring is arranged to balance the centrifugal force exerted at a speed of 500 revs. per min. when the balls are in mid-position, and hence the graphs A and B cross each other. In consequence, the force exerted by the spring is only 28·4 lb. when the sleeve is in the zero position; the spring pressure is therefore balanced by centrifugal force when the balls rotate at a speed of N revs. per min., such that,

$$\frac{1}{32 \cdot 2} \times \left(\frac{\pi N}{30}\right)^2 \times 0 \cdot 35 = 28 \cdot 4 \text{ lb.}$$

The value of N to satisfy the equation is 489. Taking values of the radius of the ball-path and the force to compress the spring from the graph B, additional values of N may be found. Hence the graph C may be obtained; from this it is evident that each position of the sleeve corresponds to a particular speed of the balls.

When the engine is not running the throttle-valve is fully open, the governor balls are as near as they can approach to the spindle, and the sleeve is in its zero position. When the engine runs with a light load the throttle-valve must be nearly closed and the sleeve will accordingly approach its highest position. As the load is increased the throttle-valve is more widely opened; since this is effected by making the sleeve descend, it follows that the greater the load the smaller will be the speed. In constructing a governor, the spring is arranged to cause the least amount of speed variation which will afford security against "hunting."

Assume that the bell-crank pivots (*C, Fig.* 158) *are* 6 *in. apart, that the arms of each bell-crank lever are of equal length, and that each ball weighs* 4 *lb. If the force acting along the spring is* 50 *lb. when the arms occupy the positions shown in Fig.* 158, *at what number of revs. per min. is the spindle rotating?*

The spring exerts a force of 25 lb. on the end of each horizontal arm. The centrifugal force acting on each ball is therefore 25 lb. Hence,

$$\text{centrifugal force} = 25 \text{ lb.} = \frac{4 \times \omega^2 \times 0{\cdot}25}{g} = 0{\cdot}0311\omega^2.$$

$$\omega^2 = \frac{25}{0{\cdot}0311} = 805.$$

$$\omega = \sqrt{805} = 28{\cdot}4 \text{ radians per sec.}$$

$$\text{Revs. per min.} = \frac{60 \times 28{\cdot}4}{2\pi} = 271.$$

Assume that at this speed, viz., 271 *revs. per min., the balls are floating in the position previously stated. Suppose that the speed increases to* 280 *revs. per min. without any change of radius of the path of the balls. What resistance is exerted by the sleeve?*

Let F be the centrifugal force at 280 revs. per min. Beyond the alteration of speed, nothing has changed, and centrifugal force being proportional to the square of the rotational speed,

$$\frac{F}{25} = \frac{(280)^2}{(271)^2} = \frac{78{,}500}{73{,}500} = 1{\cdot}07.$$

$$F = (25 \times 1{\cdot}07) \text{ lb.} = 26{\cdot}7 \text{ lb.}$$

Increase of force on each ball = 26·7 lb. — 25 lb. = 1·7 lb. The arms of the bell-crank lever being of equal length and there being two balls, the total force balanced by the resistance of the sleeve is 2 × 1·7 lb. or 3·4 lb.

EXAMPLES.

146. A wheel-rim weighs 4 tons. Its mean radius is $3\frac{1}{2}$ ft. and its sectional area is 136 sq. in. If the safe tensile stress of the metal is 2 tons per sq. in., what is the limiting angular speed of this wheel in revs. per min. ?

147. The rim of a cast-iron pulley has a mean radius of 12 in., is 6 in. broad, and $\frac{1}{2}$ in. thick ; when the pulley revolves at the rate of 150 revs. per min. what is the centrifugal force on the pulley-rim per 1 in. length of rim ?

(1 *cub. in. of cast-iron weighs* 0·26 *lb.*)

What is the tensile stress per sq. in. in the pulley-rim under these conditions ?

What is the limiting speed of rotation for this pulley if the tenacity of cast-iron is 12·5 tons per sq. in. ?

148. A jet of water having a sectional area of 2 sq. in. and a speed of 100 ft. per sec., impinges upon the buckets of a Pelton wheel. The speed of the water as it leaves the wheel is 10 ft. per sec. What is the H.P. of the wheel ?

149. A jet of water of circular section and 0·5 in. in diameter impinges tangentially at a speed of 100 ft. per sec. upon a plate curved into an arc of a circle which subtends an angle of 120° at the centre of the circle. What is the resultant pressure upon the plate ?

150. A simple governor rotates at such a speed that the cone of revolution is 9 in. high. The centre of the ball rotates in a circle of 15 in. radius, and the centrifugal force on one ball is 7 lb. Find the weight of the ball.

151. Each of the balls of a Watt governor weighs 4 lb. The arms are inclined at 30° to the vertical when the governor rotates at N revs. per min. and are each 15 in. long. Find N.

152. A simple governor is constructed with four links of equal length, much as in Fig. 156, but without the central load W, and the two lower links are jointed by a pin intersecting the axis of revolution. Normally the speed is 160 revs. per min. ; when the speed rises to 200 revs. per min. the throttle-valve is closed. What movement of the sleeve has taken place ?

153. A spring loaded governor constructed as shown in Fig. 158 has two balls weighing 4 lb. each. The arms of the bell-crank levers are of equal length. When the centre of gravity of each ball is 3 in. away from the axis of rotation the total compressive force in the spring is 50 lb. At how many revs. per min. is the governor rotating ?

154. A Porter governor (Fig. 156) has two 4 lb. balls. The links are of equal length and the axes of the top and bottom pin joints of these links intersect the axis of revolution. The central load weighs 30 lb. When the speed of rotation is 20 radians per sec., what is the height of the cone of revolution ?

ANSWERS TO EXAMPLES

The greater part of the results were found with the aid of a slide-rule.

1. 22·4 lb.
2. 2 : 3.
3. 55,652 lb.
4. 3·22 ft. per sec. per second.
5. 20·34 lb.
6. 0·287 ft. per sec. per second and 3·6 ft.
7. 0·0932 : 1 and 24 poundals.
8. 73 lb.
9. 23·4 ft.
10. 913 ft.-lb.
11. 11,200 lb.
12. 2·6 in.
13. 10,945 lb.
14. 1,021 lb. per ft.
15. 44·5 ft. per sec.
16. 38·5 ft. per sec.
17. 60,000 ft.
18. 3,030 lb.
19. 18·7 lb.
20. 8,130 lb.
21. 2·68 ft. per sec.
22. 250 dynes.
23. 13,900 lb.
24. 64,600 lb.
25. 37·4 lb.
26. 5 ft. per sec. to the left.
27. 22·2 ft. per sec.
28. B continues in the same direction at a speed of 13·1 ft. per sec.; A moves at a speed of 53·1 ft. per sec.
29. Backward at 200 ft. per sec.
30. 1,600 ft. per sec.
31. 57·6 ft.
32. 630 tons.
33. 1,930 lb.
34. 27,000 lb.
35. 20 cub. in.
36. 4,210 lb.
37. 1,275 lb.
38. 3,240 lb.
39. $\sqrt{50}$ or 7·07 ft.
40. 49 lb.
41. 200 : 203.
42. 4·4 in.
43. 36,800 lb. and 6·285 ft.
44. 187·5 lb.
45. 416 lb.
46. 0·8.
47. 17·6 in.
48. 3,720 lb.
49. 1·26.
50. 0·82 ft.
51. 144.
52. 0·0416 in.
53. 2,457 lb.
54. 927 lb. per sq. in. and 1,300 ft.-tons.
55. (*a*) 78,540 lb.; (*b*) 4·2 ft.
56. 28·5 H.P.
57. 135,600 ft.-lb.
58. 51 ft.-tons.
59. 53·3 per cent.
60. 59 lb. per min.
61. 34·6 ft.
62. 0·7.
63. 21·96 cub. ft.
64. 0·04 sq. in.
65. 800 lb.-ft.
66. 0·75 : 1.
67. Equal strength.
68. 3·485 tons.
69. 2·39 in.
70. 17·8 lb.
71. 21,300 lb.
72. 22·22 sq. in.

73. At 10 tons, 29·73 tons-ft.
At 6 tons, 63·77 tons-ft.
At 7 tons, 56·95 tons-ft.
74. 150 lb.
75. 62·5 tons-ft. and 4·5 tons.
76. $9\frac{1}{8}$ lb. on the left; $5\frac{5}{8}$ lb. on the right. Bending moment, $13\frac{1}{3}$ lb.-ft.; shearing force, $\frac{5}{8}$ lb.
77. Shearing force, 1·5 tons; bending moment, $7\frac{7}{8}$ tons-ft.
78. 0·35 in.
79. 96·3 lb.-ft.
80. 19,400 lb.-in.
81. 8,800 lb. per sq. in.
82. 662·5 lb.-ft.
83. 13·36 in.
84. 13,100 lb. per sq. in.
85. 6·93 in.

$$\frac{\text{weight of two shafts}}{\text{weight of one shaft}} = \frac{0·96}{1}$$

86. 29·8 H.P.
87. 7·175 H.P.
88. 41·4 H.P.
89. 24 I.H.P.
90. 2,260 lb.
91. 1,200 lb.
92. 15·2 H.P.
93. 34·5 I.H.P.
94. 40·5 W.H.P.
95. 0·497 W.H.P.
96. 1·31 H.P.
97. 71 per cent.
98. 1·54 H.P.
99. 0·79 H.P.
100. 1·29 H.P.
101. 26·25 lb.
102. 2·28 H.P.
103. 0·793 H.P.
104. 13·6 H.P.
105. 19·7 lb.
106. 380°.
107. 335 lb. and 95 lb.
108. 0·193.
109. 4·1 ft.
110. 20 threads per in.
111. 25.
112. 3 in.
113. 213 ft. per min.
114. 293 ft.
115. 3 : 1.
116. 6 ft.
117. 0·377 radians.
118. 3·77.
119. 0·884 ft. per sec.
120. 323 lb.
121. (*a*) 0·5 ft. per sec.; (*b*) 0·125 radians per sec.; (*c*) 4,000 lb.-ft.
122. 554 lb.-in.
123. 104 ft. per sec.
124. 0·5.
125. 346·4 lb.
126. 1·155 : 1.
127. 20°.
128. 60°.
129. 1 ft.
130. 10·9 ft. per sec. and 5·45 ft. per sec.
131 and 132. Diagrams only.
133. 4·14 sec.
134. 0·1056 ft.
135. 11·75 ft. per sec.
136. 10,240 ft. per sec. per second.
137. 0·13 ft.
138. 0·567 sec.
139. 117·5 lb.
140. 115 revs. per min.
141. 0·0174 lb.
142. 1,875 lb.
143. 150 lb.
144. 1045 lb.
145. 230 lb.
146. 600 revs. per min.
147. 5·97 lb.; 23·88 lb. per sq. in.; 5,130 revs. per min.
148. 48·5 H.P.
149. 45·8 lb.
150. 4·2 lb.
151. 52.
152. 0·08 ft.
153. 271 revs. per min.
154. 0·685 ft.

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